ELSEVIER
J. Math. Pures Appl. 82 (2003) 1–42
JOURNAL DE MATHÉMATIQUES PURES ET APPLIQUÉES
www.elsevier.com/locate/matpur

Coupled connections on a compact Riemann surface

Indranil Biswas *

School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Bombay 400005, India

Received 15 August 2002

Abstract

We consider holomorphic differential operators on a compact Riemann surface X whose symbol is an isomorphism. Such a differential operator of order n on a vector bundle E sends E to $K_X^{\otimes n} \otimes E$, where K_X is the holomorphic cotangent bundle. We classify all those holomorphic vector bundles E over X that admit such a differential operator. The space of all differential operators whose symbol is an isomorphism is in bijective correspondence with the collection of pairs consisting of a flat vector bundle E over X and a holomorphic subbundle of E satisfying a transversality condition with respect to the connection.

MSC: 32C38; 14F10

Keywords: Coupled connection; Differential operator; Grassmann embedding

1. Introduction

We will start by recalling some results of [3], the precursor to the present paper. Let X be a compact connected Riemann surface. Let $\mathcal{B}_X(1,n)$ denote the space of all isomorphism classes of differential operators of degree n, between line bundles, whose symbol is an isomorphism.

On the other hand, consider the collection of all triples of the form (V, ∇, L), where V is a holomorphic vector bundle over X of rank n equipped with a holomorphic connection ∇ and L is a line subbundle of E satisfying a transversality condition. The transversality condition in question says that the filtration

$$L := L_1 \subset L_2 \subset L_3 \subset \cdots \subset E$$

* Corresponding author.
E-mail address: indranil@math.tifr.res.in (I. Biswas).

0021-7824/02/$ – see front matter
doi:10.1016/S0021-7824(02)00004-1

of E constructed using the second fundamental form has the property that each L_i is a subbundle of E of rank i. For any such triple (V, ∇, L), the underlying vector bundle V must be isomorphic to the jet bundle $J^{n-1}(L \otimes (TX)^{\otimes n})$ [3, p. 5, Theorem 2.9].

Let $\mathcal{A}_X(1, n)$ denote the space of all isomorphism classes of triples of the above type. In [3] we constructed a bijective map between the two spaces $\mathcal{A}_X(1, n)$ and $\mathcal{B}_X(1, n)$ [3, p. 17, Theorem 4.9]. In [3], the Cartesian product

$$\mathfrak{P}(X) \times \left(\bigoplus_{i=3}^{n} H^0\big(X, K_X^{\otimes i}\big) \right),$$

where $\mathfrak{P}(X)$ is the space of all projective structures on X and K_X, is the holomorphic cotangent bundle, was identified with a certain class of holomorphic immersions of the universal cover of X into $\mathbb{CP}^{n-1}$ that are equivariant with respect to some representation of the group of deck transformations of the universal cover into $\mathrm{GL}(n, \mathbb{C})$ [3, Theorem 5.5].

Here we study the higher rank situation as opposed to the earlier situation of differential operators between line bundles and transversal line subbundles of flat bundles.

Fix a theta characteristic on X, that is, a holomorphic line bundle $\mathcal{L}$ over X with $\mathcal{L}^{\otimes 2} \cong K_X$. Fix an integer $n \geqslant 2$.

Let E be a holomorphic vector bundle of rank r over X. Consider holomorphic differential operators $D \in H^0(X, \mathrm{Diff}_X^n(E, F))$, where F is some holomorphic vector bundle over X, such that the symbol $\sigma(D) \in H^0(X, \mathrm{Hom}(E, (TX)^{\otimes n} \otimes F))$ is an isomorphism. We prove that there is such a differential operator if and only if $E \otimes \mathcal{L}^{\otimes(n-1)}$ admits a holomorphic connection (Theorem 5.1). Let $\mathcal{B}_X(r, n)$ denote the space of all equivalence classes of pairs (E, D), with $\mathrm{rank}(E) = r$ and $\sigma(D) = \mathrm{Id}_E$.

For any $i \in [0, n-1]$, let $\mathcal{K}_i \subset J^{n-1}(E)$ denote the subbundle of the $(n-1)$th order jet bundle defined by the kernel of the projection $J^{n-1}(E) \to J^{n-i-1}(E)$. Any differential operator D in $\mathcal{B}_X(r, n)$ gives a flat connection ∇^D on the jet bundle $J^{n-1}(E)$. This connection has the property that $\nabla^D(\mathcal{K}_i) = K_X \otimes \mathcal{K}_{i+1}$.

The above connections arising from differential operators led us to define coupled connections. A coupled connection is a triple of the form (V, ∇, F), where ∇ is a flat connection on a holomorphic vector bundle V over X of rank rn and $F \subset V$ a subbundle of rank r with the property that if we inductively define coherent subsheaves $\{F_i\}$ of V with $F_1 = F$ and $\nabla(F_i) = K_X \otimes F_{i+1}$, then each F_i, $i \in [1, n]$, is a subbundle of V of rank ri.

We prove that all coupled connections arise from differential operators whose symbol is an isomorphism (Theorem 4.2). Let $\mathcal{A}_X(r, n)$ denote the space of all equivalence classes of coupled connections (V, ∇, F), with $\mathrm{rank}(V) = nr$ and $\mathrm{rank}(F) = r$.

For a vector bundle E over X and any $k \geqslant 3$ define:

$$\mathcal{V}_E(k) := H^0\big(X, K_X \otimes \mathrm{End}(E)\big) \oplus H^0\big(X, K_X^{\otimes 2} \otimes \mathrm{ad}(E)\big) \oplus \bigoplus_{i=3}^{k} H^0\big(X, K_X^{\otimes i} \otimes \mathrm{End}(E)\big),$$

and define $\mathcal{V}_E(2) := H^0(X, K_X \otimes \mathrm{End}(E)) \oplus H^0(X, K_X^{\otimes 2} \otimes \mathrm{ad}(E))$.

Consider all quadruples of the form $(E, \nabla, \mathfrak{p}, v)$, where (E, ∇) is a flat vector bundle of rank r, $\mathfrak{p}$ a projective structure on X and $v \in \mathcal{V}_E(n)$. The space of all isomorphism classes of such quadruples with $\operatorname{rank}(E) = r$ is identified with $\mathcal{B}_X(r, n)$ (Theorem 6.1).

Let $\widetilde{X}$ be a universal cover of X with Γ as the group of all deck transformations. Let V a complex vector space of dimension nr. A holomorphic map γ from $\widetilde{X}$ to the Grassmannian $G(r, V)$ is called equivariant if there is a homomorphism ρ of Γ to $\mathrm{GL}(V)$ such that γ commutes with actions of Γ with Γ acting on $G(r, V)$ through ρ. Such a map γ will be called nondegenerate if it satisfies a certain transversality condition with respect to the linear subspaces in $G(r, V)$. In particular, a nondegenerate map is an immersion. The details of the definition are given in Section 8. The space of all nondegenerate equivariant maps of $\widetilde{X}$ to $G(r, V)$ is identified with $\mathcal{A}_X(r, n)$. Using this it follows that the space of all nondegenerate equivariant maps is identified with the space of all quadruples $(E, \nabla, \mathfrak{p}, v)$ of the above type (Proposition 8.3).

2. Jet bundles and holomorphic connection

In this section we will first briefly recall the basic definitions and properties of jet bundles that will be useful here.

Let X be a compact connected Riemann surface or, equivalently, an irreducible smooth projective curve over $\mathbb{C}$. The complex surface $X \times X$ will be denoted by Z. Let $\Delta \subset Z$ be the (reduced) diagonal divisor consisting of points of the form (x, x). Let $p_i : Z \to X$, $i = 1, 2$, denote the projection to the ith factor of the Cartesian product.

Let E be a holomorphic vector bundle over X. For any integer $k \geqslant 0$, the kth order *jet bundle* of E, denoted by $J^k(E)$, is defined to be the following direct image on X:

$$J^k(E) := p_{1*}\left(\frac{p_2^* E}{p_2^* E \otimes \mathcal{O}_{X\times X}(-(k+1)\Delta)}\right). \tag{2.1}$$

So $J^k(E)$ is a holomorphic vector bundle of rank $(k+1)\operatorname{rank}(E)$ over X.

Let K_X denote the holomorphic cotangent bundle of X. Take a point $x \in X$ and a holomorphic function f defined on an open subset containing x with $f(x) = 0$. Sending $(df)^{\otimes k}(x) \in (K_X^{\otimes k})_x$ to $f^k/k!$ we get a homomorphism

$$f_{\mathcal{O},k} : K_X^{\otimes k} \to J^k(\mathcal{O}_X).$$

It is easy to see that the above condition uniquely determines the homomorphism $f_{\mathcal{O},k}$.

Consider the natural inclusion of $\mathcal{O}_Z(-(k+1)\Delta)$ in $\mathcal{O}_Z(-k\Delta)$. This inclusion induces an exact sequence of vector bundles:

$$0 \to K_X^{\otimes k} \otimes E \to J^k(E) \to J^{k-1}(E) \to 0. \tag{2.2}$$

The inclusion map $K_X^{\otimes k} \otimes E \to J^k(E)$ is constructed by using the homomorphism $f_{\mathcal{O},k}$ defined above. The surjective homomorphism in (2.2) corresponds to the restriction of a holomorphic section of E defined on the kth order infinitesimal neighborhood of a point of X to the $(k-1)$th order infinitesimal neighborhood of that point.

A holomorphic section of E over an open subset U of X gives a section of $J^i(E)$ over U for each $i \geqslant 0$. This defines a homomorphism from the sheaf defined by E to the sheaf defined by $J^i(E)$. But this is not $\mathcal{O}_X$-linear unless $i = 0$.

From the above definition of a jet bundle it is immediate that any holomorphic homomorphism $E \to F$ of vector bundles over X induces a homomorphism

$$J^k(E) \to J^k(F) \tag{2.3}$$

for each $k \geqslant 0$, and the corresponding diagram of homomorphisms

$$\begin{tikzcd} J^{k+1}(E) \arrow[r] \arrow[d] & J^{k+1}(F) \arrow[d] \\ J^k(E) \arrow[r] & J^k(F) \end{tikzcd}$$

is commutative, where the vertical homomorphisms are as in (2.2).

For any of integer $l \geqslant 0$, there is a natural injective homomorphism of vector bundles

$$\tau : J^{l+1}(E) \to J^1\big(J^l(E)\big). \tag{2.4}$$

Indeed, if s is a holomorphic section of E defined over an open subset U that vanishes of order $l+2$ at some point $x_0 \in U$, then the section of $J^l(E)$ over U defined by s vanishes of order 2 at x_0. This observation immediately implies that we have a homomorphism τ as in (2.4) which is defined by the condition that for any (locally defined) holomorphic section s of E, the section of $J^{l+1}(E)$ defined by s is sent by τ to the section of $J^1(J^l(E))$ defined by s.

Since the homomorphism τ will be very useful for us, some of its properties will be mentioned here. We will describe the image of τ explicitly in the special case where $l = 1$. We have two homomorphisms,

$$h_1, h_2 : J^1\big(J^1(E)\big) \to J^1(E),$$

defined as follows. Since E is a quotient of $J^1(E)$ (as in (2.2)), using (2.3) we have a homomorphism h_1 as above. On the other hand, from (2.2) we have a projection h_2 of $J^1(J^1(E))$ to $J^1(E)$ (by setting $k = 1$ and $E = J^1(E)$ in (2.2)). Note that

$$h_1 - h_2 : J^1\big(J^1(E)\big) \to K_X \otimes E \subset J^1(E),$$

as the compositions of the projection $J^1(E) \to E$ in (2.2) with h_1 and h_2 coincide. The image of τ coincides with the kernel of the homomorphism $h_1 - h_2$ in the special case of $l = 1$. It should be emphasized that the diagram

$$\begin{tikzcd} J^1(J^{l+1}(E)) \arrow[r] \arrow[d] & J^{l+1}(E) \arrow[d, equal] \\ J^1(J^l(E)) & J^{l+1}(E) \arrow[l, "\tau"'] \end{tikzcd}$$

does not commute (unless $E = 0$ or $l = 0$); the two homomorphisms from $J^1(J^{l+1}(E))$ are defined using (2.2) and (2.3). See [8] for the details.

Let E and F be two holomorphic vector bundles over X. The sheaf of *differential operators* of order k from E to F, which is denoted by, $\mathrm{Diff}^k_X(E, F)$ is defined as:

$$\mathrm{Diff}^k_X(E, F) := \mathrm{Hom}\big(J^k(E), F\big). \tag{2.5}$$

Consider the composition:

$$\sigma : \mathrm{Diff}^k_X(E, F) = J^k(E)^* \otimes F \to \big(K_X^{\otimes k} \otimes E\big)^* \otimes F, \tag{2.6}$$

where the right-hand side homomorphism is obtained from the injective homomorphism in (2.2), is known as the *symbol map*. So we have an exact sequence of vector bundles

$$0 \to \mathrm{Diff}^{k-1}_X(E, F) \to \mathrm{Diff}^k_X(E, F) \xrightarrow{\sigma} \big(K_X^{\otimes k} \otimes E\big)^* \otimes F \to 0 \tag{2.7}$$

which is obtained from (2.2).

Now we will recall the definition of a holomorphic connection and some of its standard properties.

For a holomorphic vector bundle E over X, consider the vector bundle $\mathrm{Diff}^1_X(E, E)$. The exact sequence (2.7) becomes:

$$0 \to \mathrm{Hom}(E, E) \to \mathrm{Diff}^1_X(E, E) \xrightarrow{\sigma} TX \otimes \mathrm{Hom}(E, E) \to 0,$$

where TX denotes the holomorphic tangent bundle of X. The subbundle

$$\mathrm{At}(E) := \sigma^{-1}(TX \otimes \mathrm{Id}_E) \subset \mathrm{Diff}^1_X(E, E)$$

is known as the *Atiyah bundle* [1]. So (2.7) gives an exact sequence

$$0 \to \mathrm{End}(E) \to \mathrm{At}(E) \to TX \to 0 \tag{2.8}$$

which is known as the *Atiyah exact sequence*.

A *holomorphic connection* on E is by definition a holomorphic splitting of the Atiyah exact sequence constructed in (2.8). The curvature of a holomorphic connection is a section of $\bigwedge^2(TX)^* \otimes \mathrm{End}(E)$. Since $\dim X = 1$, any holomorphic connection over X is automatically flat.

A splitting of the Atiyah exact sequence (2.8) gives a homomorphism from TX to $\mathrm{Diff}^1_X(E,E)$. From the definition of the sheaf of differential operators given in (2.5) it follows immediately that

$$\mathrm{Diff}^1_X(E,E)\otimes K_X \cong \mathrm{Diff}^1_X(E,K_X\otimes E).$$

Using this isomorphism, a holomorphic connection on E is a first order differential operator

$$\nabla \in H^0\bigl(X,\mathrm{Diff}^1_X(E,K_X\otimes E)\bigr) \tag{2.9}$$

with the identity automorphism of E as its symbol. It is easy to see that this condition on symbol of ∇ is equivalent to the *Leibnitz identity* which says that

$$\nabla(fs) = df\otimes s + f\nabla s,$$

where s is any locally defined holomorphic section of E and f is any locally defined holomorphic function on X.

Let ∇ be a differential operator as in (2.9) whose symbol is the identity automorphism of E. The condition that $\sigma(\nabla)=\mathrm{Id}_E$ is equivalent to the condition that the homomorphism

$$\nabla : J^1(E)\to K_X\otimes E$$

corresponding to the differential operator (defined using (2.5)) gives a splitting of the exact sequence of vector bundles

$$0\to K_X\otimes E\to J^1(E)\to E\to 0$$

in (2.2). Therefore, a holomorphic connection on E is a homomorphism of vector bundles

$$\nabla : E\to J^1(E) \tag{2.10}$$

such that the composition

$$E\xrightarrow{\nabla} J^1(E)\to E$$

is the identity map of E (the projection $J^1(E)\to E$ is as in (2.2)).

Let ∇ be a holomorphic connection on E. If $\bar\partial$ is the Dolbeault operator defining the holomorphic structure of E, then $\nabla+\bar\partial$ is a flat connection on E compatible with its holomorphic structure. Conversely, the $(1,0)$-part of any flat connection on E compatible with its holomorphic structure defines a holomorphic connection on E. By a flat connection on a holomorphic vector bundle we will always mean one which is compatible with the holomorphic structure.

We will recall another way of defining a holomorphic connection which is due to A. Grothendieck. Consider the nonreduced divisor $2\Delta \subset Z := X \times X$. Restrict the vector bundle

$$E \boxtimes E^* := p_1^* E \otimes p_2^* E^*$$

to 2Δ. Note that the vector bundle $(E \boxtimes E^*)|_{\Delta}$ over Δ has a canonical automorphism defined by the identity automorphism of E. A holomorphic connection on E is a holomorphic section

$$D' \in \Gamma\big(2\Delta, (E \boxtimes E^*)|_{2\Delta}\big) \tag{2.11}$$

whose restriction to Δ coincides with the one given by the identity automorphism of E.

If s is a locally defined holomorphic section of E defined over an open subset $U \subset X$, then the contraction

$$s' := \big\langle D', p_2^* s\big\rangle$$

is a section of $p_1^* E$ over $2\Delta \cap (U \times U)$. Since s' and $p_1^* s$ coincide over Δ (as the restriction of D' to Δ is the identity automorphism), the difference

$$s' - p_1^* s \in H^0(U, K_U \otimes E|_U),$$

where $K_U = K_X|_U$, is the holomorphic cotangent bundle of U.

Sending any (locally defined) holomorphic section s to the section $s' - p_1^* s$ constructed above we get a homomorphism from the sheaf of sections of E to that of $K_X \otimes E$. This homomorphism satisfies the Leibnitz identity. Indeed, this is an immediate consequence of the condition that the restriction of D' to Δ coincides with the identity automorphism of E. In other words, the section D' over 2Δ defines a holomorphic connection in the sense defined earlier. Conversely, a holomorphic connection D in the sense defined earlier gives a section D' of $E \boxtimes E^*$ over 2Δ which is determined by the condition that $D(s)$ coincides with $s' - p_1^* s$ for every (locally defined) holomorphic section s. See the last paragraph of Section 5 for another description of this equivalence between the two definitions of a holomorphic connection.

Let F be a holomorphic subbundle of a vector bundle E equipped with a holomorphic connection ∇. The *second fundamental form* of F for the holomorphic connection ∇ on E is the composition of homomorphism of sheaves

$$F \hookrightarrow E \xrightarrow{\nabla} K_X \otimes E \xrightarrow{\mathrm{Id} \times q} K_X \otimes (E/F),$$

where the final homomorphism is defined using the natural projection

$$q : E \to \frac{E}{F}.$$

The Leibnitz identity ensures that the second fundamental form

$$F \to K_X \otimes (E/F)$$

is a homomorphism of $\mathcal{O}_X$-modules. Consequently, it defines a vector bundle homomorphism

$$S(F, \nabla) \in H^0\big(X, K_X \otimes \mathrm{Hom}(F, E/F)\big). \tag{2.12}$$

Using a homomorphism from $K_X^* \otimes F$ to E/F as in (2.12), we obtain a holomorphic subbundle of E containing F. To construct this subbundle, let:

$$T \subset \frac{E/F}{S(F, \nabla)(K_X^* \otimes F)}$$

be the torsion part of the cokernel of the homomorphism $S(F, \nabla)$. The inverse image of T for the quotient map

$$E/F \to \frac{E/F}{S(F, \nabla)(K_X^* \otimes F)}$$

is the unique subbundle of E/F of minimal rank containing the image $S(F, \nabla)(K_X^* \otimes F)$. Denoting this subbundle of E/F by F', the inverse image

$$F_2 := q^{-1}(F')$$

is a holomorphic subbundle of E containing F, where q as before denotes the quotient map. From the construction of F_2 it is immediate that F is preserved by ∇ (that is, ∇ induces a holomorphic connection on F) if and only if $F = F_2$.

Repeating this construction, by replacing F by F_2, a filtration

$$F_0 := 0 \subset F_1 := F \subsetneq F_2 \subsetneq F_3 \subsetneq \cdots \subsetneq F_{n-1} \subsetneq F_n \subseteq E \tag{2.13}$$

of E by subbundles is obtained (if $F = F_2$ then $n = 1$). Here F_{i+1} is the subbundle obtained by substituting F_i for F in the above construction of F_2. The subbundle F_n is where this iterated construction of filtration stabilizes. If F_n is a proper subbundle of E, then we will define F_{n+1} to be E itself. Note that the subbundle F_n is preserved by the connection ∇.

Consider the filtration in (2.13). Since the differential operator ∇ sends F_i to $K_X \otimes F_{i+1}$, the second fundamental forms for the subbundles in the filtration $\{F_i\}$ of E give a homomorphism of vector bundles

$$S_i : F_i/F_{i-1} \to K_X \otimes (F_{i+1}/F_i) \tag{2.14}$$

for each $i \in [1, n-1]$. So, S_1 coincides with $S(F, \nabla)$ in (2.12). It is easy to see that $S_i \neq 0$ for all $i \in [1, n-1]$. More precisely, for each $i \in [1, n-1]$, the homomorphism S_i is generically surjective, that is, the number of points of X where S_i fails to be fiberwise surjective is finite.

3. Differential operators whose symbol is an isomorphism

Let V and W be two holomorphic vector bundles over X. Let

$$D \in H^0\big(X, \mathrm{Diff}^n_X(V, W)\big)$$

be a differential operator of order n over X. So the symbol of D (defined in (2.6))

$$\sigma(D) \in H^0\big(X, \mathrm{Hom}\big(K_X^{\otimes n} \otimes V, W\big)\big),$$

where K_X is the holomorphic cotangent bundle of X.

Assume that the symbol $\sigma(D)$ is an isomorphism. In particular, the vector bundle W is holomorphically isomorphic to $K_X^{\otimes n} \otimes V$. Consider the composition

$$D' := \sigma(D)^{-1} \circ D : V \xrightarrow{D} W \xrightarrow{\sigma(D)^{-1}} K_X^{\otimes n} \otimes V$$

which is a differential operator of order n from V to $K_X^{\otimes n} \otimes V$. It is easy to see that the symbol of D' is the identity automorphism of V. This shows that we can replace W by $K_X^{\otimes n} \otimes V$ and assume that the symbol $\sigma(D)$ is the identity automorphism of V.

Let

$$D \in H^0\big(X, \mathrm{Diff}^n_X\big(V, K_X^{\otimes n} \otimes V\big)\big) \tag{3.1}$$

be a differential operator of order n, where $n \geqslant 1$, whose symbol is the identity automorphism of V.

Consider the exact sequence:

$$0 \to K_X^{\otimes n} \otimes V \to J^n(V) \to J^{n-1}(V) \to 0, \tag{3.2}$$

of vector bundles over X constructed in (2.2). The differential operator D gives a splitting of this exact sequence. To explain this splitting, first note that from the definition of a differential operator in (2.5) it follows immediately that D is a homomorphism from $J^n(V)$ to $K_X^{\otimes n} \otimes V$. Consider the composition of the inclusion of $K_X^{\otimes n} \otimes V$ in $J^n(V)$ (as in (3.2)) with this homomorphism defined by D. The condition that the symbol of D is the identity automorphism of V is equivalent to the condition that this composition homomorphism is the identity automorphism of $K_X^{\otimes n} \otimes V$. Therefore, we have a splitting of the exact sequence (3.2). The splitting defines a holomorphic homomorphism of vector bundles

$$F_D : J^{n-1}(V) \to J^n(V) \tag{3.3}$$

whose composition with the projection in (3.2) is the identity automorphism of $J^{n-1}(V)$.

Consider the commutative diagram of vector bundles:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_X^{\otimes n} \otimes V & \longrightarrow & J^n(V) & \longrightarrow & J^{n-1}(V) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow \tau & & \| & & \\ 0 & \longrightarrow & K_X \otimes J^{n-1}(V) & \longrightarrow & J^1(J^{n-1}(V)) & \longrightarrow & J^{n-1}(V) & \longrightarrow & 0, \end{array} \tag{3.4}$$

where τ is defined in (2.4) and the horizontal exact sequences are as in (2.2). The homomorphism of vector bundles

$$\tau \circ F_D : J^{n-1}(V) \to J^1\big(J^{n-1}(V)\big)$$

is a splitting of the bottom exact sequence in (3.4), where F_D is the homomorphism obtained in (3.3) from D. Let

$$\Phi_D : J^1\big(J^{n-1}(V)\big) \to K_X \otimes J^{n-1}(V)$$

be the projection homomorphism obtained from this splitting defined by $\tau \circ F_D$. Since the projection Φ_D is a splitting of the bottom exact sequence in (3.4), the corresponding differential operator

$$\Phi_D \in H^0\big(X, \mathrm{Diff}^1_X\big(J^{n-1}(V), K_X \otimes J^{n-1}(V)\big)\big)$$

has the identity automorphism of $J^{n-1}(V)$ as its symbol. As it was explained in Section 2, such a differential operator defines a holomorphic connection on $J^{n-1}(V)$.

Let $\Phi(D)$ denote the holomorphic connection on $J^{n-1}(V)$ defined by the first order differential operator Φ_D constructed above. Consider the filtration of the vector bundle $J^{n-1}(V)$,

$$\big(K_X^{\otimes(n-1)} \otimes V \hookrightarrow\big) J^{n-1}(V) \to J^{n-2}(V) \to J^{n-3}(V) \to \cdots \to J^0(V) = V, \tag{3.5}$$

obtained from (2.2). Note that all the homomorphisms $J^{i+1}(V) \to J^i(V)$ are surjective. So it is a filtration of $J^{n-1}(V)$ of length n and the subsequent quotients are isomorphic to $K_X^{\otimes i} \otimes V$ with $i \in [0, n-1]$.

On the other hand, we have the filtration constructed in (2.13) of $J^{n-1}(V)$, equipped with the holomorphic connection $\Phi(D)$, for the subbundle $K_X^{\otimes(n-1)} \otimes V \subset J^{n-1}(V)$ (as in (2.2)). The following proposition says that this filtration of $J^{n-1}(V)$ coincides with the filtration in (3.5). In particular, the filtration in (2.13) obtained by setting $E = J^{n-1}(V)$, $F = K_X^{\otimes(n-1)} \otimes V$ and $\nabla = \Phi(D)$ is independent of the choice of the differential operator D.

Proposition 3.1. *The filtration of $J^{n-1}(V)$ defined in* (3.5) *coincides with the one obtained in* (2.13) *after setting $E = J^{n-1}(V)$, $F = K_X^{\otimes(n-1)} \otimes V$ and $\nabla = \Phi(D)$*

in (2.13). *Furthermore, the homomorphism S_i defined in* (2.14) *coincides with the identity automorphism of $K_X^{\otimes(n-i)} \otimes V$, where $i \in [1, n-1]$.*

Proof. For the purpose of the proof of this proposition we will describe the second fundamental form defined in (2.12).

As in the set up of (2.12), let E be a holomorphic vector bundle over X equipped with a holomorphic connection ∇. Let F be a holomorphic subbundle of E.

In Section 2 we saw that a holomorphic connection on E is a homomorphism $E \to J^1(E)$ such that the composition in (2.10) is the identity automorphism of E. Let

$$h : E \to J^1(E)$$

be the homomorphism for the connection ∇ on E.

Consider the commutative diagram

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
 & & & F \arrow[d, "\iota"] & \\
 & & J^1(E) \arrow[d, "j"] & E \arrow[l, "h"'] \arrow[d] & \\
0 \arrow[r] & K_X \otimes (E/F) \arrow[r] & J^1(E/F) \arrow[r, "q"] \arrow[d] & E/F \arrow[r] \arrow[d, equal] & 0 \\
 & & E/F \arrow[r, equal] & E/F, &
\end{tikzcd} \tag{3.6}$$

where j is defined using the homomorphism (2.3), ι is the inclusion map, and the horizontal exact sequence is the one in (2.2); the homomorphism h was defined above.

Consider the homomorphism:

$$j \circ h \circ \iota : F \to J^1(E/F).$$

The commutativity of the diagram implies that $q \circ j \circ h \circ \iota = 0$. Consequently, $j \circ h \circ \iota$ gives a homomorphism:

$$S(F) : F \to K_X \otimes (E/F). \tag{3.7}$$

This homomorphism $S(F)$ coincides with the second fundamental form $S(F, \nabla)$ defined in (2.12).

Take any integer $i \in [0, n-1]$. Let $\mathcal{K}_i$ denote the kernel of the projection $J^{n-1}(V) \to J^{n-1-i}(V)$.

Set $E = J^{n-1}(V)$, $\nabla = \Phi(D)$ and $F = \mathcal{K}_i$ in the construction of the second fundamental form (see (2.12)). First observe that the following diagram of homomorphisms of vector bundles is commutative

$$\begin{tikzcd} J^n(V) \arrow[r] \arrow[d] & J^{n-i}(V) \arrow[d] \\ J^1(J^{n-1}(V)) \arrow[r] & J^1(J^{n-1-i}(V)), \end{tikzcd} \tag{3.8}$$

where the top horizontal homomorphism is obtained from (2.2) and the bottom horizontal homomorphism is obtained from the combination of (2.2) and (2.3); the vertical homomorphisms are as in (2.4).

Using the commutativity of the above diagram (3.8) and the earlier description of the second fundamental form (given in (3.7)) the following description of the second fundamental form $S(\mathcal{K}_i, \Phi(D))$ is obtained.

Consider the commutative diagram of vector bundles over X:

$$\begin{tikzcd} & & & 0 \arrow[d] & \\ & & & \mathcal{K}_i \arrow[d, "\iota"] & \\ & & J^n(V) \arrow[r, "p"] \arrow[d, "j"] & J^{n-1}(V) \arrow[d] & \\ 0 \arrow[r] & K_X^{\otimes(n-i)} \otimes V \arrow[r] & J^{n-i}(V) \arrow[r, "q"] \arrow[d] & J^{n-i-1}(V) \arrow[r] \arrow[d, equal] & 0 \\ & & J^{n-i-1}(V) \arrow[r, equal] & J^{n-i-1}(V), & \end{tikzcd} \tag{3.9}$$

where ι is the inclusion map and the horizontal exact sequence is the one in (2.2).

We noted that the differential operator D in (3.1) gives a splitting of the exact sequence (3.2) which is defined by the homomorphism F_D in (3.3).

Just as in the construction of the homomorphism $S(F)$ in (3.7), the composition $j \circ F_D \circ \iota$ gives a homomorphism

$$S\big(\mathcal{K}_i, \Phi(D)\big) : \mathcal{K}_i \to K_X^{\otimes(n-i)} \otimes V \tag{3.10}$$

of vector bundles, as $q \circ j \circ F_D \circ \iota = 0$. This homomorphism $S(\mathcal{K}_i, \Phi(D))$ is the second fundamental form of the subbundle $\mathcal{K}_i$ for the connection $\Phi(D)$. Indeed, this follows from the construction of the second fundamental form given in (3.7) and the commutativity of the diagram (3.8).

On the other hand, we have a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{K}_i \arrow[r] \arrow[d, "f"] & J^{n-1}(V) \arrow[r] \arrow[d] & J^{n-i-1}(V) \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & K_X^{\otimes(n-i)} \otimes V \arrow[r] & J^{n-i}(V) \arrow[r] & J^{n-i-1}(V) \arrow[r] & 0 \end{tikzcd} \tag{3.11}$$

of homomorphisms of vector bundles, where the bottom exact sequence is obtained from (2.2) (the top exact sequence is obtained from the definition of $\mathcal{K}_i$), and the vertical homomorphism in the middle is also obtained from (2.2).

It is straight-forward to check that the above homomorphism f (defined by the commutativity of the diagram in (3.11)) coincides with the homomorphism $S(\mathcal{K}_i, \Phi(D))$ constructed in (3.10).

The above homomorphism f has the property that it vanishes on the subbundle

$$\mathcal{K}_{i-1} \subset \mathcal{K}_i .$$

Furthermore, the induced homomorphism

$$\frac{\mathcal{K}_i}{\mathcal{K}_{i-1}} = K_X^{\otimes(n-i)} \otimes V \xrightarrow{f} K_X^{\otimes(n-i)} \otimes V$$

(induced by f) is the identity automorphism of $K_X^{\otimes(n-i)} \otimes V$. Since the second fundamental form of the subbundle $\mathcal{K}_i$ for the connection $\Phi(D)$ coincides with the homomorphism f, the proof of the proposition is complete. □

From Proposition 3.1 it follows that the differential operators whose symbol is an isomorphism produce connections of a very specific type. In the next section we will study the space of all connections of this special type.

4. Coupled connections

As in Section 2, let E be a holomorphic vector bundle over the Riemann surface X. Consider a pair of the form (F, ∇), where ∇ is a holomorphic connection on E and F is a holomorphic subbundle of E.

Definition 4.1. The pair (F, ∇) is called a *coupled connection* if the corresponding filtration in (2.13) has the property that $F_n = E$ (that is, the filtration does not stabilize to a proper subbundle of E), and the homomorphism S_i in (2.14) is an isomorphism for all $i \in [1, n-1]$.

If D is a differential operator as in (3.1) with the identity automorphism of V as its symbol, then Proposition 3.1 says that the pair $(K_X^{\otimes(n-1)} \otimes V, \Phi(D))$ is a coupled connection. We will show in this section that all coupled connections arise from differential operators.

Let (F, ∇) (as in Definition 4.1) be a coupled connection on E. Consider the corresponding filtration, constructed in (2.13), of the vector bundle E for the coupled connection (F, ∇). Set

$$Q := \frac{E}{F_{n-1}} \tag{4.1}$$

to be the final quotient in (2.13) (since (F, ∇) is a coupled connection, F_{n-1} is a proper subbundle of E). Let q denote the projection of E to Q.

We will now construct a homomorphism of vector bundles

$$\psi_i : E \to J^i(Q) \tag{4.2}$$

for each $i \geqslant 0$.

Take a point $x \in X$ and a vector $v \in E_x$ in the fiber of E over x. Let s_v denote the (unique) holomorphic section of E defined over a contractible open neighborhood of x satisfying the following two conditions:

(1) $s_v(x) = v$,
(2) the section s_v is flat with respect to the connection ∇ on E.

So $q(s_v)$ is a holomorphic section of Q defined around x, where q, defined above, is the projection. Now restricting $q(s_v)$ to the ith order infinitesimal neighborhood of x we get an element $\psi_i(v) \in J^i(Q)_x$ in the fiber of $J^i(Q)$ over x. The homomorphism ψ_i in (4.2) is defined by sending any v to the corresponding element $\psi_i(v)$ constructed above.

As in the proof of Proposition 3.1, for any $i \in [0, n-1]$, let $\mathcal{K}_i$ denote the kernel of the projection $J^{n-1}(Q) \to J^{n-1-i}(Q)$ defined using (2.2). Note that $\mathcal{K}_i$ is a subbundle of $\mathcal{K}_{i+1}$ and

$$\frac{\mathcal{K}_{i+1}}{\mathcal{K}_i} \cong K_X^{\otimes(n-i-1)} \otimes Q \tag{4.3}$$

which is obtained using (2.2). More precisely, the isomorphism is induced by the homomorphism f in (3.11).

Theorem 4.2. *The homomorphism*

$$\psi_{n-1} : E \to J^{n-1}(Q)$$

is an isomorphism. Furthermore,

$$\psi_{n-1}(F_i) = \mathcal{K}_i$$

for each $i \in [1, n-1]$, *where* F_i *are as in* (2.13). *In other words, there is an isomorphism of vector bundles*

$$\overline{\psi}_i : \frac{E}{F_i} \to J^{n-1-i}(Q)$$

such that the following diagram is commutative

$$\begin{array}{ccc} E & \xrightarrow{\psi_{n-1}} & J^{n-1}(Q) \\ \downarrow & & \downarrow \\ E/F_i & \xrightarrow{\overline{\psi}_i} & J^{n-1-i}(Q), \end{array}$$

where $J^{n-1}(Q) \to J^{n-1-i}(Q)$ *is the projection obtained from* (2.2).

Using the isomorphism

$$\frac{F_i}{F_{i-1}} \cong \frac{\mathcal{K}_i}{\mathcal{K}_{i-1}} \cong K_X^{\otimes(n-i)} \otimes Q$$

(*the second isomorphism is in* (4.3)) *for all* $i \in [1, n-1]$, *the isomorphism* S_i *in* (2.14) *coincides with the identity automorphism of* $K_X^{\otimes(n-i)} \otimes Q$.

Proof. From the definition of a coupled connection it follows immediately that

$$\frac{\operatorname{rank}(E)}{n} = \operatorname{rank}(Q) = \frac{\operatorname{rank}(J^{n-1}(Q))}{n}.$$

Take any point $x \in X$ and $v \in E \setminus \{0\}$ a nonzero vector in the fiber of E over x. To prove that ψ_{n-1} is an isomorphism it suffices to show that $\psi_{n-1}(v) \neq 0$.

For any vector bundle W over X, by $(W)_x$ (also by W_x) we will denote the fiber of W at x. We will prove the following assertion:

If $\psi_i(v) = 0$ for some $i \in [0, n-1]$, then

$$v \in (F_{n-i-1})_x \subset E_x,$$

where F_{n-i-1} is the subbundle of E in (2.13) and the homomorphism ψ_i is defined in (4.2).

Note that as $F_0 = 0$, setting $i = n-1$ in the above assertion we conclude that if $\psi_{n-1}(v) = 0$ then $v = 0$. Also, note that if $\psi_i(v) = 0$, then $\psi_j(v) = 0$ for all j with $0 \leqslant j \leqslant i$.

To prove the above assertion by induction on i, first assume that $\psi_0(v) = 0$. The condition that $\psi_0(v) = 0$ is clearly equivalent to the condition that the image of v under the quotient projection $E_x \to Q_x$ vanishes. Therefore, we have $v \in (F_{n-1})_x \subset E_x$, where $(F_{n-1})_x$, as before, denotes the fiber of the vector bundle F_{n-1} at x. In other words, the above assertion is valid for $i = 0$.

Now assume that $\psi_j(v) = 0$ for $j = 0, 1$. We already saw that $v \in (F_{n-1})_x$. Let

$$v_1 \in \frac{(F_{n-1})_x}{(F_{n-2})_x}$$

be the image of v by the natural projection $(F_{n-1})_x \to (F_{n-1})_x/(F_{n-2})_x$. The condition that $\psi_1(v) = 0$ implies that the image

$$S_{n-1}(v_1) \in \big(K_X \otimes (F_n/F_{n-1})\big)_x$$

vanishes, where the homomorphism S_{n-1} is the one defined in (2.14). Indeed, this is a straight-forward consequence of the definition of the second fundamental form. But S_{n-1} is an isomorphism (a condition for a coupled connection in Definition 4.1). Consequently, we have $v_1 = 0$. This implies that

$$v \in (F_{n-2})_x \subset E_x.$$

In other words, the above assertion is valid for $i = 1$.

Finally, assume that there is a nonnegative integer $k \leqslant n - 2$ such that

(1) the assertion is proved for all i with $i \leqslant k$; and
(2) for the given vector $v \in E_x$ we have $\psi_j(v) = 0$ for all j with $0 \leqslant j \leqslant k + 1$.

Since $\psi_j(v) = 0$ for all j with $0 \leqslant j \leqslant k$ and the assertion is proved for all i with $i \leqslant k$, we have:

$$v \in (F_{n-k-1})_x \subset E_x.$$

Let

$$v_k \in \frac{(F_{n-k-1})_x}{(F_{n-k-2})_x}$$

be the image of v by the natural projection $(F_{n-k-1})_x \to (F_{n-k-1})_x/(F_{n-k-2})_x$.

The condition that $\psi_{k+1}(v) = 0$ implies that the image

$$S_{n-k-1}(v_k) \in \big(K_X \otimes (F_{n-k}/F_{n-k-1})\big)_x$$

vanishes, where the homomorphism S_{n-k-1} is the one defined in (2.14). This again is a straight-forward consequence of the fact that S_{n-k-1} is the second fundamental form of the subbundle F_{n-k-1} for the connection ∇. We recall from the definition of a coupled connection that the homomorphism S_{n-k-1} is in fact an isomorphism. Consequently, we have $v_k = 0$. This implies that

$$v \in (F_{n-k-2})_x \subset E_x.$$

In other words, the above assertion is valid for $i = k + 1$.

So, by induction, the above assertion is established. We already noted that setting $i = n-1$, the assertion implies that the homomorphism ψ_{n-1} defined in (4.2) is an isomorphism.

Since $\operatorname{rank}(F_i) = \operatorname{rank}(Q)i = \operatorname{rank}(\mathcal{K}_i)$, to prove that

$$\psi_{n-1}(F_i) = \mathcal{K}_i$$

for any $i \in [0, n-1]$ it suffices to show that $\psi_{n-1}(F_i) \subseteq \mathcal{K}_i$, that is, the image of the subbundle $\psi_{n-1}(F_i) \subseteq J^{n-1}(Q)$ by the natural projection $J^{n-1}(Q) \to J^{n-1-i}(Q)$ (defined by (2.2)) vanishes.

To prove the inclusion $\psi_{n-1}(F_i) \subseteq \mathcal{K}_i$ by induction, first consider $i = n-1$. Since $J^0(Q) = Q$ and F_{n-1} is the kernel of the projection of E to its quotient Q, it follows immediately that $\psi_{n-1}(F_{n-1}) \subseteq \mathcal{K}_{n-1}$.

Now take any point $x \in X$ and any vector $v \in (F_{n-2})_x \subset E_x$. As in (2.12), let $S(F_{n-2}, \nabla)$ denote the second fundamental form of the subbundle F_{n-2} for the connection ∇. From the definition of a coupled connection it follows immediately that the image

$$S(F_{n-2}, \nabla)(F_{n-2}) \subseteq \frac{E}{F_{n-2}}$$

projects to zero in $Q := E/F_{n-2}$. Indeed, $S(F_{n-2}, \nabla)(F_{n-2})$ is contained in F_{n-1}. From the relationship between the second fundamental form and the first order jets (the relationship described in the proof of Proposition 3.1) we conclude that since $S(F_{n-2}, \nabla)(F_{n-2})$ projects to zero in Q, the image of F_{n-2} in $J^1(Q)$ vanishes. In other words, the inclusion

$$\psi_{n-1}(F_{n-2}) \subseteq \mathcal{K}_{n-2}$$

is valid.

Let $k \in [1, n-1]$. Assume that $\psi_{n-1}(F_i) \subseteq \mathcal{K}_i$ for all $i \in [k, n-1]$. To prove that

$$\psi_{n-1}(F_{k-1}) \subseteq \mathcal{K}_{k-1}$$

first note that $S(F_{k-1}, \nabla)(F_{k-1}) \subseteq F_k$, where $S(F_{k-1}, \nabla)$, as in (2.12), is the second fundamental form of the subbundle F_{k-1} for the connection ∇. By the above assumption we have $\psi_{n-1}(F_k) \subseteq \mathcal{K}_k$. These two inputs combine together to imply that $\psi_{n-1}(F_{k-1})$ projects to zero by the quotient homomorphism $J^{n-1}(Q) \to J^{n-k}(Q)$.

This completes the proof of the assertion that

$$\psi_{n-1}(F_i) = \mathcal{K}_i$$

for any $i \in [0, n-1]$.

Take a point $x \in X$ and a vector $v \in (F_i)_x$ in the fiber of F_i over x. As before, let s_v denote the (unique) flat section of E (for the connection ∇) defined around x with $s_v(x) = v$. Since $\psi_{n-1}(F_i) \subseteq \mathcal{K}_i$, the (locally defined) section $q(s_v)$ of Q vanishes of order

$n-i$ at x, where q, as before, is the projection of E to its quotient Q. The element in $(\mathcal{K}_i)_x$ defined by $q(s_v)$ coincides (by definition) with $\psi_{n-1}(v)$.

To construct $S_i(v)$, where S_i is defined in (2.14), take a holomorphic section t_v of F_i defined around x with $t_v(x) = v$. The element in $(K_X \otimes (F_{i+1}/F_i))_x$ defined by $s_v - t_v$ coincides (by definition) with $S_i(v)$. Note that $q(t_v) = 0$. Therefore, $q(s_v - t_v) = q(s_v)$. This implies that the automorphism of $K_X^{\otimes(n-i)} \otimes Q$ defined by S_i and the isomorphism

$$\frac{\mathcal{K}_i}{\mathcal{K}_{i-1}} \cong K_X^{\otimes(n-i)} \otimes Q$$

in (4.3) actually coincides with the identity map of $K_X^{\otimes(n-i)} \otimes Q$. This completes the proof of the theorem. □

Consider the homomorphism of vector bundles

$$\psi_n \circ \psi_{n-1}^{-1} : J^{n-1}(Q) \to J^n(Q) \tag{4.4}$$

for the coupled connection (F, ∇) on the vector bundle E. It is easy to see that this homomorphism $\psi_n \circ \psi_{n-1}^{-1}$ gives a splitting of the exact sequence of vector bundles

$$0 \to K_X^{\otimes n} \otimes Q \to J^n(Q) \to J^{n-1}(Q) \to 0 \tag{4.5}$$

constructed in (2.2). To prove that this is a splitting, take any point $x \in X$ and any vector $v \in E_x$. As before, let s_v denote the flat section of E defined on an open neighborhood of x with $s_v(x) = v$. Now, $\psi_{n-1}(v)$ (respectively, $\psi_n(v)$) coincides (by definition) with the restriction of $q(s_v)$ to the $(n-1)$th order (respectively, nth order) infinitesimal neighborhood of x, where q is the projection of E to Q. This immediately implies that the restriction of $\psi_n \circ \psi_{n-1}^{-1}(\psi_{n-1}(v))$ to the $(n-1)$th order infinitesimal neighborhood of x coincides with $\psi_{n-1}(v)$. In other words, the homomorphism $\psi_n \circ \psi_{n-1}^{-1}$ is a splitting of the exact sequence (4.5).

This splitting $\psi_n \circ \psi_{n-1}^{-1}$ gives a homomorphism of vector bundles

$$H_\nabla \in H^0\big(X, \operatorname{Hom}\big(J^n(Q), K_X^{\otimes n} \otimes Q\big)\big).$$

Such a homomorphism H_∇ defines a global differential operator

$$D_\nabla \in H^0\big(X, \operatorname{Diff}_X^n\big(Q, K_X^{\otimes n} \otimes Q\big)\big) \tag{4.6}$$

(see the definition (2.5)). Since the homomorphism H_∇ is obtained from a splitting of the exact sequence (4.5), the symbol of the differential operator D_∇ is the identity automorphism of Q. Indeed, this follows immediately from the definition of symbol given in (2.6).

In Proposition 3.1 we constructed a coupled connection from a differential operator whose symbol is the identity map, and now we have constructed such a differential operator, namely D_∇, starting with a coupled connection (F, ∇). We will show that these two constructions are inverses of each other. For this we first need to define when two differential operators or two connections are isomorphic.

Let V_1 and V_2 be two holomorphic vector bundles over X and

$$D_i \in H^0\big(X, \mathrm{Diff}_X^n\big(V_i, K_X^{\otimes n} \otimes V_i\big)\big),$$

$i = 1, 2$, two differential operators on X with symbol $\sigma(D_i) = \mathrm{Id}_{V_i}$.

Definition 4.3. We will call D_1 to be *equivalent* to D_2 if there is a holomorphic isomorphism

$$T : V_1 \to V_2$$

such that $D_2 \circ T = (T \otimes \mathrm{Id}_{K_X^{\otimes n}}) \circ D_1$ as differential operators (of order n) from V_1 to $K_X^{\otimes n} \otimes V_2$.

Let E_1 and E_2 be two holomorphic vector bundles over X. Let (F_1, ∇_1) and (F_2, ∇_2) be two coupled connections on E_1 and E_2, respectively.

Definition 4.4. The coupled connection (F_1, ∇_1) will be called *equivalent* to (F_2, ∇_2) if there is a holomorphic isomorphism

$$A : E_1 \to E_2$$

such that $A(F_1) = F_2$ and $\nabla_2 \circ A = (A \otimes \mathrm{Id}_{K_X}) \circ \nabla_1$ as differential operators (of order one) from E_1 to $K_X \otimes E_2$ (see (2.9)).

If the two differential operators D_1 and D_2 are equivalent, then the corresponding coupled connections (constructed in Proposition 3.1) are clearly equivalent. Conversely, if (F_1, ∇_1) and (F_2, ∇_2) are equivalent coupled connections on two vector bundles E_1 and E_2, respectively, then the corresponding differential operators (constructed above using Theorem 4.2) are equivalent.

Proposition 4.5. *Let D (respectively, (F, ∇)) be a differential operator (respectively, coupled connection) as in Proposition 3.1 (respectively, Theorem 4.2). Let (F^1, ∇^1) (respectively, D^1) denote the coupled connection (respectively, differential operator) constructed from D (respectively, (F, ∇)). Let D^2 (respectively, (F^2, ∇^2)) be the differential operator (respectively, coupled connection) constructed from (F^1, ∇^1) (respectively, D^1). Then D coincides with D^2 and the coupled connection (F, ∇) is equivalent to (F^2, ∇^2).*

Proof. That D^2 coincides with D is a straight-forward consequence of the constructions in Proposition 3.1 and Theorem 4.2. To explain this, let $\underline{J^{n-1}(V)}$ denote the locally constant complex sheaf on X defined by the (locally defined) flat sections for the connection ∇^1 on

$J^{n-1}(V)$, where V is the vector bundle over which the differential operator D is defined. The composition homomorphism

$$\psi_n \circ \psi_{n-1}^{-1} : J^{n-1}(V) \to J^n(V),$$

where $\psi_i : \underline{J^{n-1}(V)} \to J^i(V)$ is defined as in (4.2), coincides with the homomorphism obtained from the splitting of the exact sequence (3.2) defined by the differential operator D. This implies that the two differential operators

$$D, D^2 \in H^0\big(X, \mathrm{Diff}_X^n\big(V, K_X^{\otimes n} \otimes V\big)\big)$$

coincide.

As in the statement of the proposition, let (F, ∇) be a coupled connection on a vector bundle E over X. Let Q denote the final quotient of E in the filtration (2.13). So, $F \cong K_X^{\otimes(n-1)} \otimes Q$. We have:

$$D^1 \in H^0\big(X, \mathrm{Diff}_X^n\big(Q, K_X^{\otimes n} \otimes Q\big)\big)$$

and the coupled connection (F^2, ∇^2) is on the vector bundle $J^{n-1}(Q)$.

Consider the isomorphism ψ_{n-1} in Theorem 4.2. This isomorphism clearly takes the subbundle F of E to the subbundle F^2 of $J^{n-1}(Q)$. This follows from the assertion in Theorem 4.2 that $\psi_{n-1}(F_1) = \mathcal{K}_1$. More generally, the filtration (2.13) corresponding to the subbundle F for the connection ∇ is taken to the filtration (2.13) corresponding to the subbundle F^2 for the connection ∇^2. Indeed, both these filtrations coincide with the filtration in (3.5) (for $Q = V$). Using the description of the homomorphism S_i in Theorem 4.2 it follows that the isomorphism ψ_{n-1} takes the connection ∇ to ∇^2. Therefore, the two coupled connections (F, ∇) and (F^2, ∇^2) are equivalent. This completes the proof of the proposition. □

In the rest of this section we will construct examples of coupled connections.

A holomorphic vector bundle V over a compact connected Riemann surface X is called *stable* if for every proper holomorphic subbundle $W \subsetneq V$ of positive rank, the inequality

$$\frac{\mathrm{degree}(W)}{\mathrm{rank}(W)} < \frac{\mathrm{degree}(V)}{\mathrm{rank}(V)}$$

is valid [6,9].

If $\mathrm{genus}(X) > 1$, then the isomorphism classes of stable vector bundles of rank n and degree d over X are parametrized by a complex quasiprojective algebraic variety $\mathcal{M}_X(n, d)$ of dimension $n^2(\mathrm{genus}(X) - 1) + 1$. The moduli space $\mathcal{M}_X(n, d)$ is smooth and irreducible (see [9]).

Lemma 4.6. *Let X be a compact connected Riemann surface of genus at least two and V a stable vector bundle over X of rank r and degree $(1 - \mathrm{genus}(X))(n - 1)r$, with $n \geqslant 1$. Then the jet bundle $J^{n-1}(V)$ admits a holomorphic connection.*

For any holomorphic connection ∇ *on* $J^{n-1}(V)$*, the pair* $(K_X^{\otimes(n-1)} \otimes V, \nabla)$ *is a coupled connection on* $J^{n-1}(V)$*, where* $K_X^{\otimes(n-1)} \otimes V \subset J^{n-1}(V)$ *is the subbundle obtained from* (2.2) *by setting* $k = n-1$ *and* $E = V$ *in* (2.2)*. The filtration of* $J^{n-1}(V)$ *in* (2.13) *corresponding to the subbundle* $(K_X^{\otimes(n-1)} \otimes V)$ *for the connection* ∇ *on* $J^{n-1}(V)$ *coincides with the filtration constructed in* (3.5)*.*

Proof. We will use the following properties of stable vector bundles:

(1) If E is a stable vector bundle and L is a holomorphic line bundle, then $E \otimes L$ is also stable.
(2) Any stable vector bundle E is *simple*, that is, $H^0(X, \mathrm{End}(E)) = \mathbb{C}$.
(3) If E and F are stable vector bundles and

$$\frac{\mathrm{degree}(E)}{\mathrm{rank}(E)} > \frac{\mathrm{degree}(F)}{\mathrm{rank}(F)},$$

then $H^0(X, \mathrm{Hom}(E, F)) = 0$.

See [9,6] for these properties of stable vector bundles.

Fix a holomorphic line bundle $\mathcal{L}$ over X with $\mathcal{L}^{\otimes 2} \cong K_X$. In other words, $\mathcal{L}$ is a theta characteristic on X. Set

$$V_0 := V \otimes \mathcal{L}^{\otimes(n-1)}.$$

So we have $\mathrm{degree}(V_0) = 0$. Using the first of the above three properties of stable vector bundles we know that V_0 is stable.

A holomorphic vector bundle E is called *indecomposable* if $E \ncong E' \oplus E''$ for any pair of holomorphic vector bundles E' and E'' of positive rank. An indecomposable vector bundle over X of degree zero admits a holomorphic connection [1, p. 203, Proposition 19], [10].

The second one of the above three properties of stable vector bundles implies that V_0 is indecomposable. Consequently, V_0 admits a flat connection. (In fact it is known that V_0 admits a unitary flat connection; but we do not need it here.)

If E and W are two holomorphic vector bundles over X, and E admits a flat connection, then

$$J^i(E \otimes W) \cong E \otimes J^i(W) \tag{4.7}$$

for any $i \geqslant 0$. (Recall from Section 2 that by a flat connection we mean a flat holomorphic connection.) To construct an isomorphism as in (4.7), fix a flat connection ∇^E on E. For any contractible open subset $U \subset X$, any holomorphic section s of $E \otimes W$ over U must be of the form

$$s = \sum_{j=1}^{m} e_j \otimes w_j,$$

where $\{e_j\}_{j=1}^m$ is a basis of flat sections (with respect to the connection ∇^E) of E over U (so $m = \operatorname{rank}(E)$) and w_j are holomorphic sections of W over U. Once the basis $\{e_j\}_{j=1}^m$ is fixed, the sections w_j are uniquely determined. Consider the homomorphism

$$\gamma_U : J^i(E \otimes W)|_U \to \big(E \otimes J^i(W)\big)\big|_U$$

defined as follows: if s' is the section of $J^i(E \otimes W)|_U$ defined by a section s of $(E \otimes W)|_U$, then

$$\gamma_U(s') = \sum_{j=1}^m e_j \otimes w'_j,$$

where w'_j is the section of $J^i(W)|_U$ defined by the section w_j of W. It is easy to see that this defines a homomorphism γ_U of the above type. It is also straight-forward to check that this homomorphism γ_U does not depend on the choice of the basis $\{e_j\}_{j=1}^m$ of flat sections of E over U. Therefore, the locally defined homomorphisms γ_U patch together compatibly to give an isomorphism as in (4.7) over X.

Since the vector bundle V_0 defined above admits a holomorphic connection, using the isomorphism (4.7) we have:

$$J^{n-1}(V) \cong V_0 \otimes J^{n-1}\big(\big(\mathcal{L}^{\otimes(n-1)}\big)^*\big).$$

Now, since V_0 admits a holomorphic connection, to show that $J^{n-1}(V)$ admits a holomorphic connection it suffices to prove that $J^{n-1}((\mathcal{L}^{\otimes(n-1)})^*)$ admits a holomorphic connection.

It is known that $J^{n-1}((\mathcal{L}^{\otimes(n-1)})^*)$ admits a holomorphic connection [2, p. 10, Theorem 3.7]. Therefore, $J^{n-1}(V)$ admits a holomorphic connection.

Now we will show that *any* holomorphic connection on $J^{n-1}(V)$ gives a coupled connection.

We claim that there are no nonzero holomorphic homomorphisms from $\mathcal{K}_i$ to $K_X \otimes J^{n-2-i}(V)$ for any $i \in [1, n-2]$, where $\mathcal{K}_i$ as before is the kernel of the projection $J^{n-1}(V) \to J^{n-1-i}(V)$.

Note that this assertion is valid for $i = 0, n-1$, as $\mathcal{K}_0 = 0$ and $K_X \otimes J^{n-2-i}(V) = 0$ for $i = n-1$.

To prove this assertion, first note that $\mathcal{K}_i$ (respectively, $K_X \otimes J^{n-2-i}(V)$) is filtered by subbundles with the subsequent quotients being $K_X^{\otimes k} \otimes V$, where $j \in [n-i, n-1]$ (respectively, $j \in [0, n-i-1]$). This filtrations are obtained from the filtration (3.5). Since V is a stable vector bundle and

$$\operatorname{degree}\big(K_X^{\otimes\alpha} \otimes V\big) > \operatorname{degree}\big(K_X^{\otimes\beta} \otimes V\big)$$

if $\alpha > \beta$, from the first and the third of the three properties of stable vector bundles stated at the beginning of the proof it follows immediately (using the two filtrations) that

$$H^0\big(X, \operatorname{Hom}\big(\mathcal{K}_i, K_X \otimes J^{n-2-i}(V)\big)\big) = 0 \tag{4.8}$$

for any $i \in [1, n-2]$. This establishes the above assertion.

Let ∇ be any holomorphic connection on $J^{n-1}(V)$. Consider the second fundamental form $S(\mathcal{K}_i, \nabla)$ defined in (2.12). From (4.8) it follows immediately that

$$\operatorname{image}\big(S(\mathcal{K}_i, \nabla)\big) \subset K_X^{\otimes(n-i)} \otimes V \hookrightarrow K_X \otimes J^{n-1-i}(V).$$

(The inclusion homomorphism is obtained from (2.2).) Furthermore, replacing i by $i-1$ in (4.8) we get that the composition homomorphism

$$\mathcal{K}_{i-1} \hookrightarrow \mathcal{K}_i \xrightarrow{S(\mathcal{K}_i, \nabla)} K_X^{\otimes(n-i)} \otimes V$$

vanishes. These two observations together imply that the filtration constructed in (2.13) corresponding to the subbundle $\mathcal{K}_1 \subset J^{n-1}(V)$ for the connection ∇ is *finer* than the filtration of $J^{n-1}(V)$ defined by the subbundles $\{\mathcal{K}_k\}_{k=1}^{n-1}$. Consequently, the second fundamental form $S(\mathcal{K}_i, \nabla)$ is induced by an homomorphism

$$\frac{\mathcal{K}_i}{\mathcal{K}_{i-1}} \cong K_X^{\otimes(n-i)} \otimes V \to K_X \otimes \frac{\mathcal{K}_{i+1}}{\mathcal{K}_i} \cong K_X^{\otimes(n-i)} \otimes V, \tag{4.9}$$

where the isomorphisms are obtained from (4.3).

Note that

$$\begin{aligned} \deg(\mathcal{K}_k) &= k\big(1 - \operatorname{genus}(X)\big)(n-1)r + 2r \sum_{i=n-k}^{n-1} \big(\operatorname{genus}(X) - 1\big)i \\ &= rk\big(\operatorname{genus}(X) - 1\big)(n-k) \neq 0 \end{aligned}$$

for any $k \in [1, n-1]$. So $\mathcal{K}_k$ does not admit a holomorphic connection. Consequently, we have $S(\mathcal{K}_k, \nabla) \neq 0$. In other words, the homomorphism in (4.9) is not identically zero.

On the other hand, since $K_X^{\otimes(n-i)} \otimes V$ is a stable vector bundle, any nonzero endomorphism of it must be an isomorphism (the second of the three properties of stable vector bundles stated at the beginning of this proof). Therefore, the filtration constructed in (2.13) corresponding to the subbundle $\mathcal{K}_1 \subset J^{n-1}(V)$ for the connection ∇ on $J^{n-1}(V)$ coincides with the filtration of $J^{n-1}(V)$ defined by the subbundles $\{\mathcal{K}_k\}_{k=1}^{n-1}$. Since the homomorphism S_i constructed in (2.14) coincides with the homomorphism in (4.9) and this homomorphism in (4.9) is an isomorphism, we conclude that $(\mathcal{K}_1, \nabla)$ is a coupled connection. This completes the proof of the lemma. □

The converse of Lemma 4.6 is not valid. In other words, there are holomorphic vector bundles Q over X that are not stable (or even semistable) but $J^{n-1}(Q)$ admit coupled connections. In the next section we will characterize all vector bundles Q with the property that $J^{n-1}(Q)$ admits a coupled connection. Note that Theorem 4.2 implies that if E admits a coupled connection, then E is isomorphic to $J^{n-1}(Q)$ for some vector bundle Q.

5. More examples of coupled connections

We will first recall an alternative description of differential operators which will be useful in deciding when there is a differential operator

$$D \in H^0\big(X, \mathrm{Diff}^n_X\big(Q, K_X^{\otimes n} \otimes Q\big)\big)$$

with $\sigma(D) = \mathrm{Id}_Q$, where Q is any given holomorphic vector bundle over X.

As in Section 2, let $\Delta \subset Z := X \times X$ be the diagonal divisor and p_i, $i = 1, 2$, the projection of Z to the ith factor. Since Δ is an effective divisor, for any holomorphic vector bundle V over Z and any integer $i \geqslant 1$ the coherent sheaf defined by the sections of V is a subsheaf of the coherent sheaf defined by the sections of $V \otimes \mathcal{O}_Z(i\Delta)$. Consider the quotient (coherent) sheaf

$$\mathcal{F}(n) := \frac{p_2^* K_X \otimes \mathcal{O}_Z((n+1)\Delta)}{p_2^* K_X}$$

over Z. Note that $\mathcal{F}(n)$ is supported over the nonreduced divisor $(n+1)\Delta$, and it is identified with the restriction of the line bundle $p_2^* K_X \otimes \mathcal{O}_Z((n+1)\Delta)$ to $(n+1)\Delta$.

Let $U \subset X$ be an open subset and $z : U \to \mathbb{C}$ a holomorphic coordinate function on it. We have a homomorphism of sheaves

$$\gamma_U(n) : \mathcal{F}(n)|_{p_1^{-1}(U)} \to \mathcal{O}_U$$

which is defined as follows: for any section

$$s = \frac{f(z_1, z_2)}{(z_2 - z_1)^{n+1}} dz_2 \in \Gamma\big(U \times U; \mathcal{F}(n)\big)$$

over $U \times U$, where (z_1, z_2) is the coordinate function on $U \times U$ defined by:

$$(z_1, z_2)(u_1, u_2) = \big(z_1(u_1), z_2(u_2)\big) \in \mathbb{C}^2;$$

set:

$$\gamma_U(n)(s)(x) := \frac{1}{n!} \frac{\partial^n f}{\partial z_2^n}(x, x)$$

for any $x \in U$. It is straight-forward to check that this homomorphism $\gamma_U(n)$ does not depend on the choice of the coordinate function z on U. Consequently, these locally defined homomorphisms $\gamma_U(n)$ patch together compatibly to define a homomorphism

$$\gamma(n) : \mathcal{F}(n) \to \mathcal{O}_X$$

of sheaves.

Let E and F be two holomorphic vector bundles over X. Define the coherent sheaf:

$$\mathcal{F}(E, F; n) := \frac{p_1^* F \otimes p_2^*(K_X \otimes E^*) \otimes \mathcal{O}_Z((n+1)\Delta)}{p_1^* F \otimes p_2^*(K_X \otimes E^*)}$$

over Z which is again supported over $(n+1)\Delta$, and it is identified with the restriction of the vector bundle $p_1^* F \otimes p_2^*(K_X \otimes E^*) \otimes \mathcal{O}_Z((n+1)\Delta)$ to $(n+1)\Delta$.

There is a natural isomorphism

$$\mathcal{K} : H^0\big((n+1)\Delta, \mathcal{F}(E, F; n)\big) \to H^0\big(X, \mathrm{Diff}_X^n(E, F)\big). \tag{5.1}$$

To construct the isomorphism in (5.1), take any $\kappa \in H^0((n+1)\Delta, \mathcal{F}(E, F; n))$, and let u be a holomorphic section of E defined over an open subset U of X. So the contraction $\langle \kappa, p_2^* u\rangle$ is a section of $p_1^* F \otimes \mathcal{F}(n)$ over $p_1^{-1}(U)$ (the contraction is the natural pairing of E with E^*). Therefore, using the projection formula for $p_{1*} p_1^* F$ we have:

$$\gamma(n)\big(\langle \kappa, p_2^* u\rangle\big) \in \Gamma(U; F),$$

where the homomorphism $\gamma(n)$ is defined above. Finally, define the homomorphism $\mathcal{K}$ in (5.1) as

$$\mathcal{K}(\kappa)(u) = \gamma(n)\big(\langle \kappa, p_2^* u\rangle\big).$$

The homomorphism $\mathcal{K}$ constructed this way is clearly an isomorphism.

The above construction actually gives a homomorphism of vector bundles over X

$$p_{1*}\mathcal{F}(E, F; n) \to \mathrm{Diff}_X^n(E, F)$$

from the direct image. Since this homomorphism of vector bundles is fiberwise injective and the rank of the two vector bundles coincide, it must be an isomorphism. Since

$$H^0\big((n+1)\Delta, \mathcal{F}(E, F; n)\big) \cong H^0\big(X, p_{1*}\mathcal{F}(E, F; n)\big)$$

we get an identification $H^0((n+1)\Delta, \mathcal{F}(E, F; n)) \cong H^0(X, \mathrm{Diff}_X^n(E, F))$ which coincides with the homomorphism $\mathcal{K}$.

The Poincaré adjunction formula says that the restriction of the line bundle $\mathcal{O}_Z(\Delta)$ to the divisor Δ coincides with

$$N_\Delta \cong T\Delta \cong \big(p_i^* TX\big)\big|\Delta,$$

where N_Δ is the normal bundle to Δ and $T\Delta$ is the (holomorphic) tangent bundle of Δ. Now, consider the natural projection:

$$\mathcal{F}(E,F;n) := \frac{p_1^*F \otimes p_2^*(K_X \otimes E^*) \otimes \mathcal{O}_Z((n+1)\Delta)}{p_1^*F \otimes p_2^*(K_X \otimes E^*)} \\ \to \frac{p_1^*F \otimes p_2^*(K_X \otimes E^*) \otimes \mathcal{O}_Z((n+1)\Delta)}{p_1^*F \otimes p_2^*(K_X \otimes E^*) \otimes \mathcal{O}_Z(n\Delta)} \cong \operatorname{Hom}(E,F) \otimes (TX)^{\otimes n},$$

where the vector bundle $\operatorname{Hom}(E,F) \otimes (TX)^{\otimes n}$ over X is considered as the sheaf supported over the reduced diagonal Δ using the natural identification of X with Δ defined by $x \mapsto (x,x)$; the isomorphism of the quotient with $\operatorname{Hom}(E,F) \otimes (TX)^{\otimes n}$ is defined using the above identification $\mathcal{O}_Z(\Delta)|_\Delta \cong TX$. Note that this projection simply restricts the sheaf $\mathcal{F}(E,F;n)$ to the subscheme $\Delta \hookrightarrow (n+1)\Delta$. Combining this projection with the homomorphism $\mathcal{K}$ in (5.1) we get a homomorphism

$$H^0\big(X, \operatorname{Diff}_X^n(E,F)\big) \to H^0\big(X, \operatorname{Hom}(E,F) \otimes (TX)^{\otimes n}\big)$$

which coincides with the symbol homomorphism σ defined in (2.6).

Fix a theta characteristic $\mathcal{L}$ over X, that is, $\mathcal{L}^{\otimes 2} \cong K_X$. Let Q be a holomorphic vector bundle of rank r over X.

Theorem 5.1. *There is a differential operator*

$$D \in H^0\big(X, \operatorname{Diff}_X^n\big(Q, K_X^{\otimes n} \otimes Q\big)\big)$$

with $\sigma(D) = \mathrm{Id}_Q$ if and only if the vector bundle $Q \otimes \mathcal{L}^{\otimes(n-1)}$ admits a holomorphic connection.

Proof. Let D be a differential operator over X of the above type whose symbol is the identity automorphism of Q. Using the isomorphism $\mathcal{K}$ in (5.1) the operator D gives a section

$$\kappa \in H^0\big((n+1)\Delta, \mathcal{F}\big(Q, K_X^{\otimes n} \otimes Q; n\big)\big).$$

The condition $\sigma(D) = \mathrm{Id}_Q$ is equivalent to the condition that the restriction of the section κ to $\Delta \hookrightarrow (n+1)\Delta$ is the identity automorphism of Q.

Set

$$Q_0 := Q \otimes \mathcal{L}^{\otimes(n-1)}.$$

Note that

$$p_1^*\big(K_X^{\otimes n} \otimes Q\big) \otimes p_2^*(K_X \otimes Q^*) \otimes \mathcal{O}_Z\big((n+1)\Delta\big) \\ \cong p_1^*Q_0 \otimes p_2^*Q_0^* \otimes \big(p_1^*\mathcal{L}^{\otimes(n+1)} \otimes p_2^*\mathcal{L}^{\otimes(n+1)} \otimes \mathcal{O}_Z\big((n+1)\Delta\big)\big)$$

as $\mathcal{L}^{\otimes 2} \cong K_X$. Now, the restriction of the line bundle

$$\mathcal{L}_1 := p_1^*\mathcal{L}^{\otimes(n+1)} \otimes p_2^*\mathcal{L}^{\otimes(n+1)} \otimes \mathcal{O}_Z\big((n+1)\Delta\big) \tag{5.2}$$

to the nonreduced divisor 2Δ has a natural trivialization [4, p. 688, Theorem 2.2] (see [4, p. 689] for a construction of this trivialization). Let s denote the trivialization of $\mathcal{L}_1^*$ over 2Δ obtained from this trivialization of $\mathcal{L}_1$.

Consider the restriction of κ to $2\Delta \hookrightarrow (n+1)\Delta$. So the contraction

$$\kappa|_{2\Delta} \otimes s \in H^0\big(2\Delta,\, p_1^* Q_0 \otimes p_2^* Q_0^*\big)$$

with its restriction to Δ being the identity automorphism of Q_0 (follows from the condition that $\sigma(D) = \mathrm{Id}_Q$).

In (2.11) of Section 2 we saw that such a section of $p_1^* Q_0 \otimes p_2^* Q_0^*$ over 2Δ gives a holomorphic connection on Q_0. Therefore, existence of a differential operator D as in the statement of the theorem implies that the vector bundle Q_0 admits a holomorphic connection.

To prove the converse, let Q be a vector bundle over X that has the property that

$$Q_0 := Q \otimes \mathcal{L}^{\otimes(n-1)}$$

admits a holomorphic connection. Fix a holomorphic connection ∇ on Q_0.

The connection ∇ gives an isomorphism of $p_1^* Q_0$ with $p_2^* Q_0$ over an analytically open neighborhood of Δ in Z. To explain this trivialization, let $U \subset X$ be a contractible open subset. Using the connection ∇, the restriction of Q_0 to U is canonically trivialized in the sense that given any pair of points $x, y \in U$, the fiber E_x is identified with E_y. This identification is done using parallel transport. Since U is contractible and ∇ is flat, the isomorphism is independent of the choice of path in U used for parallel transport.

This trivialization of $Q_0|_U$ clearly gives a isomorphism of $(p_1^* Q_0)|_{U\times U}$ with $(p_2^* Q_0)|_{U\times U}$ that restricts to the identity automorphism on Δ. Let F_U denote this isomorphism. If $V \subset U$ is a contractible open subset, then the corresponding homomorphism F_V over $V \times V$ clearly coincides with the restriction of F_U to $V \times V$. From this it follows that we have an isomorphism of $p_1^* Q_0$ with $p_2^* Q_0$ over the open subset of Z obtained by taking the union of all open subsets of the form $U \times U$ with U contractible. This union contains the diagonal Δ. The restriction of this isomorphism of $p_1^* Q_0$ with $p_2^* Q_0$ to 2Δ coincides with one given by the section of $p_1^* Q_0 \otimes p_2^* Q_0^*$ over 2Δ corresponding to the connection ∇. (In Section 2 we saw that the space of all holomorphic connections on Q_0 are in bijective correspondence with the space of all sections of $p_1^* Q_0 \otimes p_2^* Q_0^*$ over 2Δ that restrict to the identity automorphism over Δ.)

The isomorphism of $p_1^* Q_0$ with $p_2^* Q_0$ over an open neighborhood of Δ can also be described as follows. The flat connection ∇ on Q_0 induces the flat connection $p_1^*\nabla \otimes \mathrm{Id} + \mathrm{Id} \otimes p_1^*\nabla^*$ on $p_1^* Q_0 \otimes p_2^* Q_0^*$. The identity automorphism of Q_0 gives a flat section of $p_1^* Q_0 \otimes p_2^* Q_0^*$ over Δ. Now using the flat connection on $p_1^* Q_0 \otimes p_2^* Q_0^*$, this flat section over Δ has a canonical extension to any contractible neighborhood of Δ. This extension is determined by the condition that it is flat with respect to the connection on $p_1^* Q_0 \otimes p_2^* Q_0^*$.

For any $i \geqslant 1$, let

$$\Phi_i \in H^0\big(i\Delta,\, p_1^* Q_0 \otimes p_2^* Q_0^*\big) \tag{5.3}$$

be the restriction to $i\Delta$ of the isomorphism $p_1^* Q_0$ with $p_2^* Q_0$. As it was noted above, the section Φ_2 over 2Δ coincides with the section corresponding to the connection ∇ (by the correspondence in Section 2).

Consider the line bundle $\mathcal{L}_1$ over $X \times X$ defined in (5.2). Note that the restriction of $\mathcal{L}_1$ to Δ is naturally identified with the trivial line bundle (follows immediately from the Poincaré adjunction formula). The line bundle $\mathcal{L}_1$ admits a trivialization over any $i\Delta$ [4, p. 688, Theorem 2.2]. Let s be a section of $\mathcal{L}_1$ over $(n+1)\Delta$ giving a trivialization and having the further property that the restriction of s to Δ is the constant function 1 (using the identification with the trivial line bundle over Δ).

So, $\Phi_{n+1} \otimes s$ is a section over $(n+1)\Delta$ of the vector bundle

$$p_1^*\big(K_X^{\otimes n} \otimes Q\big) \otimes p_2^*(K_X \otimes Q^*) \otimes \mathcal{O}_Z\big((n+1)\Delta\big),$$

where Φ_{n+1} is defined in (5.3). Now, using the isomorphism $\mathcal{K}$ in (5.1) the section $\Phi_{n+1} \otimes s$ gives a differential operator

$$D \in H^0\big(X, \mathrm{Diff}_X^n\big(Q, K_X^{\otimes n} \otimes Q\big)\big)$$

with $\sigma(D) = \mathrm{Id}_Q$. Indeed, as Φ_1 is the identity automorphism of Q_0 and the restriction of s to Δ is the constant function 1, it follows immediately that $\sigma(D) = \mathrm{Id}_Q$. This completes the proof of the theorem. $\square$

If Q is a holomorphic vector bundle over X such that $Q_0 := Q \otimes \mathcal{L}^{\otimes(n-1)}$ admits a holomorphic connection, then using Proposition 3.1 and Theorem 5.1 it follows that the jet bundle $J^{n-1}(Q)$ admits a coupled connection. Conversely, if (F, ∇) is a coupled connection on a holomorphic vector bundle E, then $F \otimes (\mathcal{L}^{\otimes(n-1)})^*$ admits a holomorphic connection.

Using the isomorphism in (5.1), the exterior derivative

$$d : \mathcal{O}_X \to K_X$$

(which is a first order differential operator) gives an element

$$\kappa_d \in H^0\big(2\Delta, p_1^* K_X \otimes p_2^* K_X \otimes \mathcal{O}_Z(2\Delta)\big).$$

For any holomorphic connection

$$D' \in \Gamma\big(2\Delta, (E \boxtimes E^*)|_{2\Delta}\big)$$

on E as in (2.11), using the isomorphism in (5.1) the tensor product

$$D' \otimes \kappa_d \in H^0\big(2\Delta, p_1^*(K_X \otimes E) \otimes p_2^*(K_X \otimes E^*) \otimes \mathcal{O}_Z(2\Delta)\big)$$

gives a differential operator

$$D \in H^0\big(X, \mathrm{Diff}_X^1(E, K_X \otimes E)\big)$$

with $\sigma(D) = \mathrm{Id}_E$. In other words, D is a holomorphic connection on E. This immediately gives an equivalence between the two definitions of a holomorphic connection given in Section 2.

6. Parameter space for all coupled connections

Let X be a compact connected Riemann surface.

Let $\mathcal{B}_X(r,n)$ denote the space of all equivalence classes of pairs of the form (E, D), where E is a holomorphic vector bundle of rank r over X and

$$D \in H^0\big(X, \mathrm{Diff}_X^n\big(E, K_X^{\otimes n} \otimes E\big)\big)$$

with $\sigma(D) = \mathrm{Id}_E$. (The notion of equivalence is described in Definition 4.3.)

Let $\mathcal{A}_X(r,n)$ denote the space of all equivalence classes of triples of the form (E, F, ∇), where E is holomorphic vector bundle over X of rank $(n-1)r$, ∇ a holomorphic connection on E, and $F \subset E$ a subbundle of rank r such that (F, ∇) is a coupled connection. (The notion of equivalence is described in Definition 4.4.)

In Proposition 4.5 it was shown that for each $n \geqslant 2$, the two spaces $\mathcal{A}_X(r,n)$ and $\mathcal{B}_X(r,n)$ are in bijective correspondence.

In this section we will describe the space $\mathcal{B}_X(r,n)$ more explicitly. For that purpose we need to recall the notion of a projective structure on a Riemann surface.

A Möbius transformation is an automorphism of $\mathbb{CP}^1$, that is, a function of the form $z \mapsto (az+b)/(cz+d)$, where $a, b, c, d \in \mathbb{C}$ are constants with $ad - bc = 1$. A projective structure on X is defined by giving a covering of X by holomorphic coordinate charts $\{U_i, \phi_i\}_{i \in I}$, where

$$\phi_i : U_i \to \mathbb{CP}^1$$

is a biholomorphism onto the image and $\bigcup_{i \in I} U_i = X$, such that each transition function $\phi_j \circ \phi_i^{-1}$ coincides with the restriction of some Möbius transformation to $\phi_i(U_i \cap U_j)$. Two such coverings $\{U_i, \phi_i\}_{i \in I}$ and $\{U_j, \phi_j\}_{j \in J}$ are called *equivalent* if the coordinate covering $\{U_i, \phi_i\}_{i \in I \cup J}$ obtained by taking their union also satisfies the above condition that all the transition functions are Möbius transformations. A *projective structure* on X is an equivalence class of such coverings. See [5] for the details.

The space of all projective structures on X is nonempty. It is in fact an affine space for $H^0(X, K_X^{\otimes 2})$ [5].

Fix a theta characteristic $\mathcal{L}$ on X. Consider the line bundle $\mathcal{L}_1$ over $X \times X$ defined in (5.2) that depends on an integer n. Giving a projective structure on X is equivalent to giving a trivialization of the line bundle $\mathcal{L}_1$ over 3Δ (for some $n \geqslant 0$) that restricts to the canonical trivialization of $\mathcal{L}_1$ over 2Δ. See [4, p. 688, Theorem 2.2] for the details.

Using the isomorphism $\mathcal{K}$ in (5.1), such a trivialization of

$$\mathcal{L}_1 = p_1^*\mathcal{L}^{\otimes 3} \otimes p_2^*\mathcal{L}^{\otimes 3} \otimes \mathcal{O}_Z(3\Delta)$$

(so $n = 2$) over 3Δ gives a differential operator

$$D \in H^0\big(X, \mathrm{Diff}^2_X\big(\mathcal{L}^*, \mathcal{L}^{\otimes 3}\big)\big)$$

with $\sigma(D) = 1$. From this observation it follows that giving a projective structure on X is equivalent to giving a differential operator

$$D \in H^0\big(X, \mathrm{Diff}^2_X\big(\mathcal{L}^*, \mathcal{L}^{\otimes 3}\big)\big)$$

with $\sigma(D) = 1$. Using Proposition 4.5 it follows that giving a projective structure on X is equivalent to giving a holomorphic connection on the rank two vector bundle $J^1(\mathcal{L}^*)$. From Lemma 4.6 it follows that any connection on $J^1(\mathcal{L}^*)$ gives a coupled connection provided genus$(X) \neq 1$.

For any holomorphic vector bundle E over X of rank r, let

$$\mathrm{ad}(E) \subset \mathrm{End}(E)$$

be the subbundle of rank $r^2 - 1$ defined by the trace zero endomorphisms. For any integer $k \geqslant 3$, let $\mathcal{V}_E(k)$ denote the direct sum

$$\begin{aligned}\mathcal{V}_E(k) :=\ & H^0\big(X, K_X \otimes \mathrm{End}(E)\big) \oplus H^0\big(X, K_X^{\otimes 2} \otimes \mathrm{ad}(E)\big) \\ & \oplus \bigoplus_{i=3}^{k} H^0\big(X, K_X^{\otimes i} \otimes \mathrm{End}(E)\big),\end{aligned} \tag{6.1}$$

and set $\mathcal{V}_E(2) := H^0(X, K_X \otimes \mathrm{End}(E)) \oplus H^0(X, K_X^{\otimes 2} \otimes \mathrm{ad}(E))$.

Let (E_1, ∇_1) and (E_2, ∇_2) be two holomorphic vector bundles over X equipped with holomorphic connections. They are called equivalent if there is a holomorphic isomorphism of E_1 with E_2 that takes ∇_1 to ∇_2.

For any $n \geqslant 2$, let $\mathcal{C}_X(r, n)$ denote the space of all quadruples of the form $(E, \nabla, \mathfrak{p}, v)$, where (E, ∇) is an equivalence class of vector bundles of rank r with holomorphic connection, $\mathfrak{p}$ a projective structure on X and

$$v \in \mathcal{V}_E(n)$$

with $\mathcal{V}_E(n)$ defined in (6.1).

Theorem 6.1. *The two space $\mathcal{C}_X(r, n)$ and $\mathcal{B}_X(r, n)$ are canonically bijective.*

Proof. Take any differential operator

$$D \in H^0\big(X, \mathrm{Diff}^n_X\big(E, K_X^{\otimes n} \otimes E\big)\big) \in \mathcal{B}_X(r, n).$$

So, $\sigma(D) = \mathrm{Id}_E$. Set

$$E_0 := E \otimes \mathcal{L}^{\otimes (n-1)}, \tag{6.2}$$

where $\mathcal{L}$, as before, is the theta characteristic that has been fixed. We saw in the proof of Theorem 5.1 that the differential operator D induces a holomorphic connection on E_0. Let ∇ denote the holomorphic connection on E_0 obtained from D.

Let

$$f_D : J^n(E) \to K_X^{\otimes n} \otimes E$$

be the homomorphism corresponding to D (obtained from the definition (2.5)). Using ∇ and (4.7) we have an isomorphism

$$J^n(E) \cong E_0 \otimes J^n\big(\big(\mathcal{L}^{\otimes(n-1)}\big)^*\big).$$

Therefore, f_D gives a homomorphism of vector bundles

$$\psi_D : E_0 \otimes J^n\big(\big(\mathcal{L}^{\otimes(n-1)}\big)^*\big) \to K_X^{\otimes n} \otimes E.$$

By the definition (2.5) and (6.2)

$$\psi_D \in H^0\big(X, \mathrm{Diff}_X^n\big(\big(\mathcal{L}^{\otimes(n-1)}\big)^*, \mathcal{L}^{\otimes(n+1)}\big) \otimes \mathrm{End}(E_0)\big). \tag{6.3}$$

Now, consider the homomorphism of vector bundles

$$\mathrm{tr} : \mathrm{End}(E_0) \to \mathcal{O}_X$$

defined by $T \mapsto \mathrm{trace}(T)/r$, where $r = \mathrm{rank}(E_0)$. Define:

$$\mathrm{Id} \otimes \mathrm{tr} : \mathrm{Diff}_X^n\big(\big(\mathcal{L}^{\otimes(n-1)}\big)^*, \mathcal{L}^{\otimes(n+1)}\big) \otimes \mathrm{End}(E_0) \to \mathrm{Diff}_X^n\big(\big(\mathcal{L}^{\otimes(n-1)}\big)^*, \mathcal{L}^{\otimes(n+1)}\big)$$

and set

$$D_0 := (\mathrm{Id} \otimes \mathrm{tr})(\psi_D) \in H^0\big(X, \mathrm{Diff}_X^n\big(\big(\mathcal{L}^{\otimes(n-1)}\big)^*, \mathcal{L}^{\otimes(n+1)}\big)\big)$$

with ψ_D as in (6.3). Since $\sigma(D) = \mathrm{Id}_E$, it follows immediately that $\sigma(D_0) = 1$.

Since $\sigma(D_0) = 1$ and $n \geqslant 2$, the differential operator D_0 gives a projective structure on X. To describe this projective structure given by D_0, let

$$s \in H^0\big((n+1)\Delta, p^*\mathcal{L}_1^{\otimes(n+1)} \otimes p^*\mathcal{L}_2^{\otimes(n+1)} \otimes \mathcal{O}_{X\times X}\big((n+1)\Delta\big)\big)$$

be the section corresponding to D_0 obtained using the isomorphism $\mathcal{K}$ constructed in (5.1). Let

$$f : X \times X \to X \times X$$

be the involution defined by $(x, y) \mapsto (y, x)$. Restricting the section $s \otimes f^*s$ to 3Δ we get a section of the line bundle

$$p^*\mathcal{L}_1^{\otimes 2(n+1)} \otimes p^*\mathcal{L}_2^{\otimes 2(n+1)} \otimes \mathcal{O}_{X\times X}\big(2(n+1)\Delta\big)$$

over 3Δ whose restriction to 2Δ coincides with the canonical trivialization. Therefore, $s \otimes f^*s$ defines a projective structure on X [4].

The projective structure on X constructed this way from D will be denoted by $\mathfrak{p}(D)$.

Given any open subset U of X, restricting this projective structure $\mathfrak{p}(D)$ to U we get a projective structure on U. This projective structure on U will also be denoted by $\mathfrak{p}(D)$.

Using the projective structure $\mathfrak{p}(D)$ on X, the vector bundle $\mathrm{Diff}^n_X((\mathcal{L}^{\otimes(n-1)})^*, \mathcal{L}^{\otimes(n+1)})$ decomposes as

$$\mathrm{Diff}^n_X\big(\big(\mathcal{L}^{\otimes(n-1)}\big)^*, \mathcal{L}^{\otimes(n+1)}\big) \cong \bigoplus_{j=0}^{n} K_X^{\otimes j} \tag{6.4}$$

[2, p. 20, Corollary 6.6]. Note that although the Corollary 6.6 of [2] is stated only for a compact Riemann surface, it is valid for any Riemann surface, as the Theorem 6.3 of [2, p. 19] is proved (and stated) for any Riemann surface equipped with a projective structure.

Using the above isomorphism,

$$\psi_D \in \bigoplus_{j=0}^{n} H^0\big(X, K_X^{\otimes j} \otimes \mathrm{End}(E_0)\big) \cong \bigoplus_{j=0}^{n} H^0\big(X, K_X^{\otimes j} \otimes \mathrm{End}(E)\big),$$

where ψ_D is constructed in (6.3). The component of ψ_D in $H^0(X, K_X^{\otimes j} \otimes \mathrm{End}(E_0))$ will be denoted by ψ_D^j. So,

$$\psi_D = \sum_{j=0}^{n} \psi_D^j.$$

The condition that $\sigma(D) = \mathrm{Id}_E$ translates into the condition that ψ_D^0 is the identity automorphism of E_0.

Let $v(D) \in \mathcal{V}_{E_0}(n)$ be the projection of ψ_D. So

$$v(D) = \psi_D^1 + \overline{\psi_D^1} + \sum_{j=3}^{n} \psi_D^j,$$

where $\overline{\psi_D^1} \in H^0(X, K_X^{\otimes 2} \otimes \mathrm{ad}(E_0))$ is the projection of $\psi_D^1 \in H^0(X, K_X^{\otimes 2} \otimes \mathrm{End}(E_0))$ defined using the obvious projection of $\mathrm{End}(E_0)$ to $\mathrm{ad}(E_0)$.

Let

$$F : \mathcal{B}_X(r,n) \to \mathcal{C}_X(r,n) \tag{6.5}$$

be the map that sends any differential operator $D \in \mathcal{B}_X(r,n)$ to

$$\big(E_0, \nabla, \mathfrak{p}(D), v(D)\big) \in \mathcal{C}_X(r,n)$$

constructed above from D. From the construction of the map F it is immediate that F is injective. We will show that F is also surjective.

To construct the inverse of F in (6.5), take any point:

$$(E, \nabla, \mathfrak{p}, v) \in \mathcal{C}_X(r, n).$$

We saw in the proof of Theorem 5.1 that a holomorphic connection on E gives an isomorphism of $p_1^* E$ with $p_2^* E$ over an analytic neighborhood of the diagonal Δ in $X \times X$. Let

$$s(\nabla) \in H^0\big((n+1)\Delta, p_1^* E \otimes p_2^* E^*\big)$$

be the section over $(n+1)\Delta$ obtained by restricting this isomorphism. So $s(\nabla)$ is Φ_{n+1} in (5.3).

On the other hand, the projective structure $\mathfrak{p}$ gives a section

$$s(\mathfrak{p}) \in H^0\big((n+1)\Delta, p_1^* \mathcal{L}^{\otimes(n+1)} \otimes p_2^* \mathcal{L}^{\otimes(n+1)} \otimes \mathcal{O}_{X\times X}\big((n+1)\Delta\big)\big)$$

over $(n+1)\Delta$ [4, p. 688, Theorem 2.2]. Therefore, $s(\nabla) \otimes s(\mathfrak{p})$ gives a differential operator

$$D(\nabla, \mathfrak{p}) \in H^0\big(X, \mathrm{Diff}_X^n(E', K_X^{\otimes n} \otimes E')\big) = H^0\big((n+1)\Delta, \mathcal{F}\big(E', K_X^{\otimes n} \otimes E'; n\big)\big) \quad (6.6)$$

using the isomorphism $\mathcal{K}$ in (5.1), where $E' := E \otimes (\mathcal{L}^{\otimes(n-1)})^*$. Since the restriction of $s(\nabla) \otimes s(\mathfrak{p})$ to Δ is the identity automorphism of E, the symbol $\sigma(D(\nabla, \mathfrak{p}))$ coincides with the identity automorphism of E'.

Using the connection ∇ on E we have an isomorphism

$$J^k\big(\big(\mathcal{L}^{\otimes(n-1)}\big)^* \otimes E\big) \cong \big(J^k\big(\mathcal{L}^{\otimes(n-1)}\big)^*\big) \otimes E \quad (6.7)$$

for any $k \geqslant 0$ (see (4.7)). Consequently,

$$\mathrm{Diff}_X^k\big(\big(\mathcal{L}^{\otimes(n-1)}\big)^* \otimes E, \mathcal{L}^{\otimes(n+1)} \otimes E\big) \cong \mathrm{Diff}_X^k\big(\big(\mathcal{L}^{\otimes(n-1)}\big)^*, \mathcal{L}^{\otimes(n+1)}\big) \otimes \mathrm{End}(E).$$

Therefore, using the isomorphism in (6.4) corresponding to the given projective structure $\mathfrak{p}$ on X, the element

$$v \in \mathcal{V}_E(n)$$

gives a differential operator

$$D(v) \in H^0\big(X, \mathrm{Diff}_X^{n-1}\big(\big(\mathcal{L}^{\otimes(n-1)}\big)^* \otimes E, \mathcal{L}^{\otimes(n+1)} \otimes E\big)\big).$$

Since v does not have a nonzero component in $H^0(X, \mathrm{End}(E))$, the degree of the differential operator is at most $n-1$.

Using the isomorphism in (6.7) and the definition of E' we have:

$$H^0\big(X, \mathrm{Diff}_X^{n-1}\big(\big(\mathcal{L}^{\otimes(n-1)}\big)^* \otimes E, \mathcal{L}^{\otimes(n+1)} \otimes E\big)\big) \cong H^0\big(X, \mathrm{Diff}_X^{n-1}\big(E', K_X^{\otimes n} \otimes E'\big)\big).$$

Let $D(v)' \in H^0(X, \mathrm{Diff}_X^{n-1}(E', K_X^{\otimes n} \otimes E'))$ be the differential operator given by $D(v)$ using this isomorphism.

Finally define:

$$D(\nabla, \mathfrak{p}, v) := D(\nabla, \mathfrak{p}) + D(v)' \in H^0\big(X, \mathrm{Diff}_X^n\big(E', K_X^{\otimes n} \otimes E'\big)\big),$$

where $D(\nabla, \mathfrak{p})$ is constructed in (6.6).

The map

$$\mathcal{C}_X(r, n) \to \mathcal{B}_X(r, n)$$

that sends a point $(E, \nabla, \mathfrak{p}, v) \in \mathcal{C}_X(r, n)$ to $D(\nabla, \mathfrak{p}, v) \in \mathcal{B}_X(r, n)$ is the inverse of the map F constructed in (6.5). That this map is the inverse of F is a straight-forward consequence of its construction and that of F. This completes the proof of the theorem. $\square$

In the next section we will describe involutions on the spaces $\mathcal{A}_X(r, n)$, $\mathcal{B}_X(r, n)$ and $\mathcal{C}_X(r, n)$ that are compatible with the identifications between them.

7. Compatible involutions

Take any

$$D \in H^0\big(X, \mathrm{Diff}_X^n\big(E, K_X^{\otimes n} \otimes E\big)\big) \in \mathcal{B}_X(r, n).$$

Set $E_0 := E \otimes \mathcal{L}^{\otimes(n-1)}$. Using the isomorphism $\mathcal{K}$ in (5.1), we have:

$$\begin{aligned} &H^0\big(X, \mathrm{Diff}_X^n\big(E, K_X^{\otimes n} \otimes E\big)\big) \\ &\quad \cong H^0\big((n+1)\Delta, p_1^*\big(\mathcal{L}^{\otimes(n+1)} \otimes E_0\big) \otimes p_2^*\big(\mathcal{L}^{\otimes(n+1)} \otimes E_0^*\big) \otimes \mathcal{O}_{X\times X}\big((n+1)\Delta\big)\big). \end{aligned}$$

Now, let

$$\zeta : X \times X \to X \times X \tag{7.1}$$

be the involution defined by $(x, y) \mapsto (y, x)$.

If

$$s(D) \in H^0\big((n+1)\Delta, p_1^*\big(\mathcal{L}^{\otimes(n+1)} \otimes E_0\big) \otimes p_2^*\big(\mathcal{L}^{\otimes(n+1)} \otimes E_0^*\big) \otimes \mathcal{O}_{X\times X}\big((n+1)\Delta\big)\big)$$

is the section corresponding to the operator D, then consider the pullback section

$$\zeta_E^* s(D) \in H^0\big((n+1)\Delta, p_1^*\big(\mathcal{L}^{\otimes(n+1)} \otimes E_0^*\big) \otimes p_2^*\big(\mathcal{L}^{\otimes(n+1)} \otimes E_0\big) \otimes \mathcal{O}_{X\times X}\big((n+1)\Delta\big)\big),$$

where ζ is defined in (7.1). Since

$$H^0\big((n+1)\Delta,\, p_1^*\big(\mathcal{L}^{\otimes(n+1)} \otimes E_0^*\big) \otimes p_2^*\big(\mathcal{L}^{\otimes(n+1)} \otimes E_0\big) \otimes \mathcal{O}_{X\times X}\big((n+1)\Delta\big)\big)$$
$$\cong H^0\big(X, \mathrm{Diff}_X^n\big(E^*, K_X^{\otimes n} \otimes E^*\big)\big)$$

(the isomorphism constructed in (5.1)), the section $\zeta_E^* s(D)$ defines a differential operator

$$D' \in H^0\big(X, \mathrm{Diff}_X^n\big(E^*, K_X^{\otimes n} \otimes E^*\big)\big) \in \mathcal{B}_X(r,n).$$

Clearly the symbol of D' coincides with the symbol of D by using the isomorphism $\mathrm{End}(E) \cong \mathrm{End}(E^*)$. If we replace D by D' in the above construction, then the resulting differential operator coincides with D (as ζ is an involution).

Therefore, we have constructed an involution of $\mathcal{B}_X(r,n)$ that sends any D to the differential operator D' constructed above.

This involution can also be directly described. Let α and β be holomorphic sections of E and E^* respectively over some analytic open subset U of X. The differential operator D' constructed above from D is uniquely determined by the identity

$$\big\langle D(\alpha), \beta\big\rangle = \big\langle \alpha, D'(\beta)\big\rangle \in H^0\big(U, K_X^{\otimes n}\big|_U\big)$$

for all α and β, where $\langle -,-\rangle$ is the contraction of E with E^*. Note that from the above identity it follows immediately that the self-map of $\mathcal{B}_X(r,n)$ defined by $D \to D'$ is an involution.

To construct the involution on $\mathcal{A}_X(r,n)$ we will first recall a property of the second fundamental form.

Let E be a holomorphic vector bundle over X equipped with a holomorphic connection ∇ and F a holomorphic subbundle of E. So we have the second fundamental form

$$S(F,\nabla) \in H^0\big(X, K_X \otimes \mathrm{Hom}(F, E/F)\big).$$

as in (2.12). Let ∇^* denote the connection on the dual vector bundle E^* induced by ∇. Consider the subbundle $F' := (E/F)^* \subset E^*$. The second fundamental form

$$S(F',\nabla^*) \in H^0\big(X, K_X \otimes \mathrm{Hom}(F', E^*/F')\big)$$

coincides with $S(F,\nabla)$ using the obvious identification

$$\mathrm{Hom}(F, E/F) \cong \mathrm{Hom}(F', E^*/F').$$

Note that,

$$\mathrm{Hom}(F, E/F) \cong F^* \otimes (E/F) \cong \mathrm{Hom}(F', E^*/F').$$

Let (F,∇) be a coupled connection on a vector bundle E. Let

$$F_0 := 0 \subset F_1 := F \subsetneq F_2 \subsetneq F_3 \subsetneq \cdots \subsetneq F_{n-1} \subsetneq F_n = E$$

be the corresponding filtration as in (2.13).

Consider the vector bundle E^* equipped with the connection ∇^* induced by ∇. From the above observation on the second fundamental form it follows immediately that the pair $((E/F_{n-1})^*, \nabla^*)$ is a coupled connection on E^*. Since the connection on $(E^*)^* \cong E$ induced by ∇^* coincides with ∇, it follows immediately that if we replace the coupled connection (F, ∇) by $((E/F_{n-1})^*, \nabla^*)$ in the above construction, then the resulting coupled connection coincides with (F, ∇).

Consequently, we have an involution of $\mathcal{A}_X(r,n)$ that sends any coupled connection (F, ∇) on E to the coupled connection $((E/F_{n-1})^*, \nabla^*)$ on E^*.

Consider the filtration (2.13) of E for the coupled connection (F, ∇) on E. The filtration of E^* induced by this filtration of E coincides with the filtration for the coupled connection $((E/F_{n-1})^*, \nabla^*)$ on E^*. In other words, if $\{F_i\}_{i=1}^n$ (respectively, $\{F_i'\}_{i=1}^n$) is the filtration of E (respectively, E^*) constructed in (2.13) for the subbundle F (respectively, $(E/F_{n-1})^*$) for the connection ∇ (respectively, ∇^*) on E (respectively, E^*), then $F_i' = (E/F_{n-i})^*$.

The identification of $\mathcal{A}_X(r,n)$ with $\mathcal{B}_X(r,n)$ constructed in Proposition 4.5 takes this involution of $\mathcal{A}_X(r,n)$ to the involution of $\mathcal{B}_X(r,n)$ constructed earlier.

To construct the involution of $\mathcal{C}_X(r,n)$, take any point:

$$(E, \nabla, \mathfrak{p}, v) \in \mathcal{C}_X(r,n).$$

Recall that

$$v \in H^0\big(X, K_X \otimes \mathrm{End}(E)\big) \oplus H^0\big(X, K_X^{\otimes 2} \otimes \mathrm{ad}(E)\big) \oplus \bigoplus_{i=3}^{n} H^0\big(X, K_X^{\otimes i} \otimes \mathrm{End}(E)\big).$$

As in the proof of Theorem 6.1, the component of v in $H^0(X, K_X^{\otimes j} \otimes \mathrm{End}(E))$ will be denoted by v^j.

Using the isomorphisms $\mathrm{ad}(E) \cong \mathrm{ad}(E^*)$ and $\mathrm{End}(E) \cong \mathrm{End}(E^*)$ we have:

$$\hat{v} \in H^0\big(X, K_X \otimes \mathrm{End}(E^*)\big) \oplus H^0\big(X, K_X^{\otimes 2} \otimes \mathrm{ad}(E^*)\big) \oplus \bigoplus_{i=3}^{k} H^0\big(X, K_X^{\otimes i} \otimes \mathrm{End}(E^*)\big)$$

defined by the condition

$$\hat{v}^j = (-1)^j v^j.$$

Let ∇^* denote the holomorphic connection on E^* induced by ∇.

Now we have an involution of $\mathcal{C}_X(r,n)$ that sends any $(E, \nabla, \mathfrak{p}, v)$ to $(E^*, \nabla^*, \mathfrak{p}, \hat{v})$. By the identification of $\mathcal{C}_X(r,n)$ with $\mathcal{B}_X(r,n)$ constructed in Theorem 6.1, this involution of $\mathcal{C}_X(r,n)$ is taken to the involution of $\mathcal{B}_X(r,n)$ constructed earlier.

In the next section we will show that the space $\mathcal{C}_X(r,n)$ parametrizes a certain class of equivariant immersions of the universal cover of X into a Grassmannian.

8. Equivariant immersions into a Grassmannian

Fix a pair of integers $r \geqslant 1$ and $n \geqslant 2$. Let V denote a complex vector space of dimension nr. Let $\mathrm{Gr} := G(r, V)$ be the Grassmannian of all r dimensional linear subspaces of V. The tautological vector bundle of rank r over Gr will be denoted by $\overline{F}$. Let $\overline{V}$ denote the trivial vector bundle over Gr with fiber V. The quotient vector bundle $\overline{V}/\overline{F}$ over Gr will be denoted by $\overline{Q}$.

Take a universal cover

$$\pi : \widetilde{X} \to X \tag{8.1}$$

of the Riemann surface X. Let

$$\gamma : \widetilde{X} \to \mathrm{Gr} \tag{8.2}$$

be a holomorphic map.

The holomorphic tangent bundle $T\mathrm{Gr}$ is canonically identified with $\mathrm{Hom}(\overline{F}, \overline{Q})$. Therefore, the differential $d\gamma$ of the above map γ gives a homomorphism:

$$d\gamma : T\widetilde{X} \otimes \gamma^*\overline{F} \to \gamma^*\overline{Q}. \tag{8.3}$$

Assume that the homomorphism of vector bundles $d\gamma$ in (8.3) is fiberwise injective. Therefore, the image of $d\gamma$ is a subbundle of $\gamma^*\overline{Q}$ of rank r.

Let $\overline{Q}_2$ denote the quotient vector bundle $\gamma^*\overline{Q}/\mathrm{image}(d\gamma)$ over $\widetilde{X}$. Since the homomorphism $d\gamma$ is injective, the rank of $\overline{Q}_2$ is $(n-2)r$.

Therefore, for any point $y \in \widetilde{X}$, the kernel of the composition homomorphism

$$V \to (\gamma^*\overline{Q})_y \to (\overline{Q}_2)_y \tag{8.4}$$

is a subspace of V of dimension $2r$. Consequently, we have a holomorphic map

$$\gamma_2 : \widetilde{X} \to G(2r, V)$$

to the Grassmannian of $2r$-dimensional subspaces of V that sends any point $y \in \widetilde{X}$ to the kernel of the composition in (8.4).

Let $\overline{F}_2$ denote the tautological vector bundle of rank $2r$ over $G(2r, V)$. Consider the differential $d\gamma_2$ of the map γ_2 constructed above. From the description of the holomorphic tangent bundle of $G(2r, V)$ it follows (just as in (8.3)) that $d\gamma_2$ gives a homomorphism

$$d\gamma_2 : T\widetilde{X} \otimes \gamma_2^*\overline{F}_2 \to \frac{\mathcal{V}}{\gamma_2^*\overline{F}_2}, \tag{8.5}$$

where $\mathcal{V}$ is the trivial vector bundle over $\widetilde{X}$ with V as its fiber.

Note that $\gamma^*\overline{F}$ is a subbundle of $\gamma_2^*\overline{F}_2$. Let ι denote the inclusion map of $\gamma^*\overline{F}$ in $\gamma_2^*\overline{F}_2$. We have a commutative diagram of vector bundles

$$\begin{array}{ccc} T\widetilde{X}\otimes\gamma^*\overline{F} & \xrightarrow{d\gamma} & \mathcal{V}/\gamma^*F \\ \Big\downarrow{\scriptstyle \mathrm{Id}\otimes\iota} & & \Big\downarrow \\ T\widetilde{X}\otimes\gamma_2^*\overline{F}_2 & \xrightarrow{d\gamma_2} & \mathcal{V}/\gamma_2^*\overline{F}_2 \end{array} \tag{8.6}$$

over $\widetilde{X}$, where $d\gamma$ and $d\gamma_2$ are defined in (8.5) and (8.3), and the right-hand side vertical map is induced by ι. Since

$$d\gamma\big(T\widetilde{X}\otimes\gamma^*\overline{F}\big)\subset\gamma_2^*\overline{F}_2/\gamma^*\overline{F},$$

from the diagram (8.6) we have a homomorphism of vector bundles

$$d'\gamma_2 : T\widetilde{X}\otimes\frac{\gamma_2^*\overline{F}_2}{\gamma^*\overline{F}}\to\frac{\mathcal{V}}{\gamma_2^*\overline{F}_2} \tag{8.7}$$

over $\widetilde{X}$.

If $n\geqslant 3$, then assume that the homomorphism of vector bundles $d'\gamma_2$ in (8.7) is fiberwise injective. In other words, the image of $d'\gamma_2$ is a subbundle of $\mathcal{V}/\gamma^*\overline{F}_2$ of rank r.

Consequently, we have a holomorphic map

$$\gamma_3 : \widetilde{X}\to G(3r,V)$$

to the Grassmannian of $3r$ dimensional subspaces of V that sends any point $y\in\widetilde{X}$ to the kernel of the composition

$$V\to(\mathcal{V}/\gamma^*\overline{F}_2)_y\to(\mathcal{V}/\gamma^*\overline{F}_2)_y/\mathrm{image}\big(d'\gamma_2(y)\big),$$

where $d'\gamma_2$ is constructed in (8.7).

We can now inductively iterate this construction (together with the assumption on the homomorphisms being fiberwise injective) and get a filtration

$$F_1:=\gamma^*\overline{F}\subset F_2:=\gamma_2^*\overline{F}_2\subset F_3\subset\cdots\subset F_{n-1}\subset F_n:=\mathcal{V} \tag{8.8}$$

of subbundles of the vector bundle $\mathcal{V}$ over $\widetilde{X}$. So F_i is a subbundle of $\mathcal{V}$ of rank ir. The details of this construction can be found in [7]. We will describe the above filtration (8.8) using the second fundamental form.

The vector bundle $\mathcal{V}$ over $\widetilde{X}$ being a trivialized bundle is equipped with a natural holomorphic connection. The homomorphism $d\gamma$ defined in (8.3) is the second fundamental form for the subbundle $\gamma^*\overline{F}$ of the flat vector bundle $\mathcal{V}$. Indeed, this follows immediately from the fact that the canonical isomorphism

$$TG(m, V) \cong \mathrm{Hom}(\overline{F}, \overline{V}),$$

where $\overline{F}$ is the tautological bundle of rank m over the Grassmannian $G(m, V)$ and $\overline{V}$ is the trivial vector bundle over $G(m, V)$ with fiber V, is induced by the second fundamental form $S(\overline{F}, \nabla^V)$ for the subbundle $\overline{F} \subset \overline{V}$ for the connection ∇^V on $\overline{V}$ induced by its trivialization. The filtration of $\mathcal{V}$ constructed in (8.8) is simply the filtration constructed in (2.13) for the subbundle $\gamma^*\overline{F}$ of the flat vector bundle $\mathcal{V}$. The iterated assumption in the construction of (8.8) that the homomorphism

$$T\widetilde{X} \otimes (F_i/F_{i-1}) \to F_{i+1}/F_i$$

is an isomorphism is equivalent to the assumption that each homomorphism S_i, $i \in [1, n-1]$, constructed in (2.14) is an isomorphism.

Definition 8.1. A holomorphic map γ as in (8.2) will be called *nondegenerate* if the iterated assumption holds in the construction of (8.8). In other words, each S_i, $i \in [1, n-1]$, constructed in (2.14) for the subbundle $\gamma^*\overline{F}$ of the trivial vector bundle $\mathcal{V}$ is an isomorphism.

Note that if the map γ is nondegenerate, then the differential $d\gamma$ in (8.3) vanishes nowhere. This implies that any nondegenerate map is an immersion.

The group $\mathrm{GL}(V)$ of linear automorphisms of V acts on Gr in an obvious way. The actions factors through the projective group $\mathrm{PGL}(V)$. In fact, the group of all holomorphic automorphisms of Gr coincides with $\mathrm{PGL}(V)$. Let

$$\Gamma \subset \mathrm{Aut}(\widetilde{X})$$

be the group of all deck transformations for the covering π in (8.1).

Definition 8.2. A holomorphic map γ from $\widetilde{X}$ to Gr (as in (8.2)) will be called *equivariant* if there exists a homomorphism of groups

$$\rho : \Gamma \to \mathrm{GL}(V)$$

such that the following diagram of maps is commutative

$$\begin{array}{ccc}
\widetilde{X} \times \Gamma & \xrightarrow{\gamma \times \rho} & \mathrm{Gr} \times \mathrm{GL}(V) \\
\downarrow & & \downarrow \\
\widetilde{X} & \xrightarrow{\gamma} & \mathrm{Gr},
\end{array}$$

where the vertical arrows are the group actions (recall that Γ and $\mathrm{GL}(V)$ act on $\widetilde{X}$ and Gr, respectively).

The map γ in the above definition will be called equivariant with respect to ρ.

Let $\gamma : \widetilde{X} \to \mathrm{Gr}$ be an equivariant and nondegenerate map.

The action of $\mathrm{GL}(V)$ on Gr clearly lifts to the vector bundle $\overline{V}$ preserving the subbundle $\overline{F}$. Note that the total space of $\overline{V}$ is $\mathrm{Gr} \times V$. The action on $\mathrm{GL}(V)$ on the factor V in the Cartesian product is the standard one. Therefore, the condition of equivariance on the map γ implies that the two vector bundles

$$\mathcal{V} = \gamma^* \overline{V}$$

and $\gamma^* \overline{F}$ on $\widetilde{X}$ descend to X. They descend as quotients by the action of Γ. In other words, $\gamma^* \overline{F}/\Gamma$ and $\mathcal{V}/\Gamma$ are vector bundles on X.

Let E denote the descend of $\gamma^* \overline{V}$. Let F be the subbundle of E which is the descend of $\gamma^* \overline{F}$.

Since the trivial connection on $\overline{V}$ is preserved by the action of $\mathrm{GL}(V)$ on $\overline{V}$, the vector bundle E is equipped with a flat connection. We will denote this connection ∇. If γ is equivariant with respect to the representation ρ, then the monodromy representation of the connection on E is conjugate to ρ. The nondegeneracy condition of γ immediately implies that the pair (∇, F) is a coupled connection on E.

Conversely, given a vector bundle E of rank nr and a coupled connection (∇, F) on it with $r = \mathrm{rank}(F)$, consider the vector bundle $\pi^* E$ on $\widetilde{X}$ equipped with the holomorphic connection $\pi^* \nabla$. Since $\widetilde{X}$ is simply connected, any holomorphic connection on $\widetilde{X}$ is trivial. Fixing an isomorphism of a fiber of $\pi^* E$ with the vector space V, the vector bundle $\pi^* E$ gets identified with $\mathcal{V}$. Let

$$\gamma : \widetilde{X} \to \mathrm{Gr}$$

denote the holomorphic map that sends any point $y \in \widetilde{X}$ to the subspace $(\pi^* F)_y \subset (\pi^* E)_y = V$. Clearly γ is equivariant for the monodromy representation of the connection ∇. The condition on (∇, F) that the homomorphism S_i in (2.14) is an isomorphism for each $i \in [1, n-1]$ implies that the map γ is nondegenerate.

Another equivariant and nondegenerate map

$$\delta : \widetilde{X} \to \mathrm{Gr}$$

will be called *equivalent* to γ if there is an automorphism $T \in \mathrm{GL}(V)$ such that

(1) $T \circ \gamma = \delta$ for the linear action of T on Gr;
(2) the vector bundle F over X to which $\gamma^* \overline{F}$ descends to isomorphic to the vector bundle over X to which $\delta^* \overline{F}$ descends.

Let F_δ denote the vector bundle over X to which $\delta^* \overline{F}$ (in the above definition) descends. The first condition $T \circ \gamma = \delta$ implies that F and F_δ are projectively isomorphic, that is, there is a holomorphic line bundle L over X such that $F_\delta \cong F \otimes L$.

Recall the spaces $\mathcal{A}_X(r,n)$, $\mathcal{B}_X(r,n)$ and $\mathcal{C}_X(r,n)$ defined in Section 6. They all are in bijective correspondence by Proposition 4.5 and Theorem 6.1. We saw above that the space of all equivalence classes of equivariant, nondegenerate holomorphic maps of $\widetilde{X}$ to Gr is identified with $\mathcal{A}_X(r,n)$. Consequently, the following proposition follows from Proposition 4.5 and Theorem 6.1.

Proposition 8.3. *The space of all equivalence classes of equivariant nondegenerate maps from $\widetilde{X}$ to* Gr *is identified with $\mathcal{C}_X(r,n)$.*

In Section 7, we described involutions on $\mathcal{A}_X(r,n)$, $\mathcal{B}_X(r,n)$ and $\mathcal{C}_X(r,n)$ that are compatible with the identifications between them. We will describe the corresponding involution on the space of all equivalence classes of equivariant and nondegenerate maps from $\widetilde{X}$ to Gr.

If

$$\gamma : \widetilde{X} \to \mathrm{Gr}$$

is an equivariant and nondegenerate map, then consider the filtration of the trivial vector bundle $\mathcal{V}$ over $\widetilde{X}$ constructed in (8.8). Let

$$(\mathcal{V}/F_{n-1})^* \subset \mathcal{V}^*$$

be the subbundle of the dual vector bundle. This subbundle defines a map

$$\hat{\gamma} : \widetilde{X} \to G(r, V^*)$$

that sends any point $y \in \widetilde{X}$ to the subspace of V^* defined by $((\mathcal{V}/F_{n-1})^*)_y$. (The vector bundle $\mathcal{V}^*$ is trivial with fiber V^*.)

It is straight-forward to check that the map $\hat{\gamma}$ is both equivariant and nondegenerate. More precisely, if γ is equivariant with respect to the representation

$$\rho : \Gamma \to \mathrm{GL}(V),$$

then $\hat{\gamma}$ is equivariant with respect to the dual representation

$$\rho^* : \Gamma \to \mathrm{GL}(V^*).$$

The filtration of $\mathcal{V}$ for γ constructed in (8.8) coincides with the dual filtration of the filtration in (8.8) of $\mathcal{V}^*$ for $\hat{\gamma}$. In other words, if

$$F_1' := \hat{\gamma}^* \overline{F} \subset F_2' \subset \cdots \subset F_{n-1}' \subset F_n' := \mathcal{V}^*$$

is the filtration in (8.8) for $\hat{\gamma}$, then $F_i' \cong (\mathcal{V}/F_{n-i})^*$.

References

[1] M.F. Atiyah, Complex analytic connections in fibre bundles, Trans. Amer. Math. Soc. 85 (1957) 181–207.
[2] I. Biswas, Differential operators on complex manifolds with a flat projective connection, J. Math. Pures Appl. 78 (1999) 1–26.
[3] I. Biswas, Invariants for a class of equivariant immersions of the universal cover of a compact Riemann surface into a projective space, J. Math. Pures Appl. 79 (2000) 1–20.
[4] I. Biswas, A.K. Raina, Projective structures on a Riemann surface, II, Internat. Math. Res. Notices 13 (1999) 685–716.
[5] R.C. Gunning, Lectures on Riemann Surfaces, in: Math. Notes, Vol. 2, Princeton Univ. Press, Princeton, NJ, 1966.
[6] S. Kobayashi, Differential Geometry of Complex Vector Bundles, in: Math. Soc. Japan, Vol. 15, Iwanami Shoten Publishers and Princeton Univ. Press, 1987.
[7] D. Perkinson, Curves in Grassmannians, Trans. Amer. Math. Soc. 347 (1995) 3179–3246.
[8] W.F. Pohl, Differential geometry of higher order, Topology 1 (1962) 169–211.
[9] C.S. Seshadri, Fibrés vectoriels sur les courbes algébriques, Astérisque 96 (1982).
[10] A. Weil, Généralisation des fonctions abéliennes, J. Math. Pures Appl. 17 (1938) 47–87.